ABSTRACT

The present project highlights the behavior of the unsteady heat transfer phenomenon developing along a horizontal surface in terms of both Strouhal and Prandtl numbers. Based on the changes that occur in the governing equation of the studied problem, an adequate analytical law of the velocity is proposed to solve unsteady momentum equation. This result presented a good agreement with Rayleigh’s exact solution and numerical solutions of Blasius and Williams–Rhyne for different values of Strouhal numbers adopted in this study. The obtained velocity expression is included in the unsteady energy equation in order to establish the temperature profile for all considered Strouhal and Prandtl numbers. Taking into account the existence of the velocity-temperature coupling in the heat boundary layer equation, the proposed formula is used to solve unsteady energy equation for all Strouhal and Prandtl numbers. As the main results, a new analytic expression of the local heat transfer coefficient for all Strouhal and Prandtl numbers is established and interesting curves are plotted to explain the heat transfer evolutions from diffusion flow to the convective flow.
surface. Results indicate that the flow regime varies from Raleigh's diffusif flow to Blasius' convective flow, and its solution is presented in a temporal polynomial. Watkins [4] examined the case of unsteady heat transfer developed near an impulsively motion of a horizontal surface via numerical description for many Prandtl numbers values.

Hall [5] and Dennis [6] used a numerical method to solve the unsteady dynamic equations of a surface suddenly in motion. In their studies, curves of the velocity variations and the shear stress values are illustrated in case of initial and steady flows. They also found that the decay to the steady flow is in exponential way. In the same wake, Williams and Rhyne [7] studied theoretically the unsteady laminar flow regime on a sudden start of the wedge surface. They used their own new similarity transformations and solved numerically the equations governing the dynamic phenomenon. The authors denoted that the boundary layer separation is far from being for a value less than or equal to -0.091 m. By using Williams-Rhyne's transformations and homotopy analysis method (HAM), Liao [8] was able to establish an analytical function of the shear stress rate valid for a wide time over a sudden start of a horizontal surface. Hang et al. [9] was interested in studying the case of electric fluid flow over a surface exposed to both sudden movement and thermal shock. The problematic was solved by HAM and given accurate polynomial solutions results of all physical quantities looked for. Simon et al. [10] studied theoretically heat transfer evolution over suddenly movement of the wall. An analytic solution is obtained for the concurrent variations of the energy and dynamic equations governing the unsteady phenomenon for wide values of m parameter. In the work of Revnic et al. [11], the dynamic and heat transfer boundary layers developed on a stretched surface are examined. They obtained an ordinary equation from partial differential equations by using an appropriate transformation. The main result outlines an enhancement of the heat rate with the increase of Prandtl number. Zheng and Ghate [12] studied unsteady flow and found an adequate solution along a horizontal wall by employing dimensionless parameters. The conclusions provided a general solution that was applicable to different types of flows. Hafidzuddin et al. [13] used an appropriate similarity variable in order to resolve momentum and energy equations on a permeable stretching/shrinking surface problem. Via an appropriate transformation, authors changed governing equations in to simple equations and they demonstrated the stability of the obtained solution. Recently, an analytic approach is used by Nagler [14] to carry out a study of the sudden motion of a cylindrical shape. The obtained velocity profile is a polynomial in high order and it is in good agreement with the literature results. To solve the motion equation along blunt body, Bulgakov et al. [15] employed Paulhasen approach with appropriate variables in aim to calculate different thicknesses of the heat transfer and momentum boundary layers.

It appears difficult, for this problem, to come up with an accurate analytic solution for all Strouhal numbers. As a complementary investigation [16], an adequate analytic approach of the unsteady heat transfer along a horizontal wall for all Prandtl and Strouhal numbers over all time evolution which sweeping all regime flows from unsteady “diffusion flow” to steady “convective flow”.

MATHEMATICAL FORMULATIONS

Consider a longitudinal flow over a horizontal wall. The fluid is at a cold temperature $T_\infty$ and the plate is maintained at a hot temperature $T_w$, see Figure 1. The phenomenon is a boundary layer problem and the laws to be approached are the governing equations of the phenomenon. From $t\geq0$, the thermal and dynamic transfer rates begin to manifest in a dynamic and thermal boundary layer where an initials and boundary conditions are mentioned in the system (4).

\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \alpha \frac{\partial^2 T}{\partial y^2}
\end{align}

(1)

(2)

(3)

Where $T$, $t$, $\nu$ and $\alpha$ denote the temperature profile, the time, the kinematic viscosity and the thermal diffusion coefficient, respectively.

$t=0$:

$u(x, y, 0) = v(x, y, 0) = 0$ and $T(x, y, 0) = T_\infty$

$t > 0$:

$u(x, 0, t) = v(x, 0, t) = 0$ and $T(x, 0, t) = T_\infty$

$u(x, y \to \infty, t) = U_\infty$ and $T(x, y \to \infty, t) = T_\infty$

It should be noted that the flow regime is governed mainly by the Strouhal number, $St_x = \frac{x}{U_\infty}$. For $St_x >> 1$, the flow is diffusive and it is called Rayleigh flow. Whereas small values of $St_x$, the flux is convective and it is known as Blasius flow.

By using a scaling analysis well suited to this flow type, Williams and Rhyne [7] found a new variable valid over the entire time interval and verifies well the similarity variable of Rayleigh “diffusion flow” and Blasius “convective flow”:
η and ξ are the similarity variables. Whereψ is the stream function, F(η,ξ) is the stream function and θ(η,ξ) is the temperature profile.

Let’s put Eq. (5) in Eqs. (1), (2) and (3), we get

\[ (6) \]

The boundary conditions Eq. (3) reduce to

\[ \eta \rightarrow 0: F(0,x) = F_\eta(0,x) = 0 \quad \text{and} \quad \theta(0,\xi) = 1 \]

\[ \eta \rightarrow \infty: F_\eta(\infty,x) = 1 \quad \text{and} \quad \theta(\infty,\xi) = 0 \quad (8) \]

### Analytical Approach

#### Equation of motion

To know the Strouhal number impact on the flow regime, the equation of motion was solved by an approximate method using an adequate analytical function of the unsteady velocity profile. The boundary conditions are checked throughout the domain 0 ≤ η < ∞. The precision of the expression obtained from the formula will be checked by evaluating the average residual deviation Δε of Eq. (6).

When \( St_x \gg 1 \), Equation. (6) transforms to the type of Rayleigh equation and its exact solution is written as: \( F_\eta(\eta,0) = \text{erf}\left(\frac{\eta}{2}\right) \). And When \( St_x \ll 1 \) (\( t \rightarrow \infty \)), Eq. (6) suits the equation of Blasius.

It can be seen from Equation (6), if ξ augments in the range from 0 to 1, the velocity profile \( F_\eta(\infty,\xi) \) evolves from the initial flow “Rayleigh’s solution” until the steady flow “Blasius’ solution”. From mathematical and Phenomenologically point of view, when \( St_x \gg 1 \) the diffusion flow differs is essentially diverse at \( St_x < 1 \) when the flow is convective. In our previous work Bachiri and Bouabdallah [16], we proposed an adequate formula of the profile of the non-dimensional velocity taking into account the boundary-layer evolution from diffusion flows to convective flows. The analytical approach permits us to give analytical expressions of both velocity profile and skin friction coefficient on the surface in all time and for all Strouhal numbers. So, the solution is expressed as:

- For the velocity profile:

\[ F_\eta(\eta,\xi) = \text{erf}\left(\frac{\eta}{2}\exp(-0.53007\xi^{-1})\right) + \xi^\frac{1}{2} \exp(-0.1394\xi^2) \quad (9) \]

- For the skin friction coefficient on the surface:

\[ C_f = 2 \text{Re}_x^{-1/2} \xi^{-1/2} \left( \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2} \xi^{y/1} \right) \right) \quad (10) \]

\( Re_x = U_x/x/\nu \) is the local Reynolds number

#### Energy equation

Thereafter, the analytical approximation of the unsteady heat transfer is made on the assumption that the plate is brought instantaneously up to a constant temperature \( T_w \) by imposing a suitable heat flow.

When \( St_x \gg 1 \), the general equation reduces to

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (11) \]

with \( T(0, t) = T_w \) and \( T(\infty, t) = 0 \).

Eq. (11) is independent of the velocity profile and expresses the diffusion heat transfer.

If we pose \( \eta^* = -\frac{\eta}{\sqrt{\alpha t}} \), the exact solution is

\[ T(\eta^*) = \text{erfc}(\eta^*) = 1 - \text{erf}(\eta^*) \quad (12) \]

Based on the Rayleigh’s solution of the diffusion flow, the ratio thicknesses of energy and dynamic boundary layers can be expressed as follows:

\[ \delta_{\text{diffusion}} \delta y = \sqrt{\alpha t} = \text{Pr}^{1/2} \quad (13) \]

So we can evaluate the Nusselt number in the transition regime at \( y = 0 \)

\[ Nu = -\chi \frac{\partial T}{\partial y} \bigg|_{y=0} \quad (14) \]

Based on the Eq. of (12), we find

\[ Nu_0 = Pr^{1/2} \frac{\chi}{\sqrt{\alpha t}} \quad (15) \]

As we know that \( \frac{\chi}{\sqrt{\alpha t}} = \text{Re}_x^{1/2} St_x^{1/2} \),

The local Nusselt number expression in just the transient case between the wall and the fluid current is given by the formula
We note that the Nusselt number have maximum values when \((t=0^+)\), \(St_x \gg 1\). However, over time, the \(Nu\) number decreases in a similar way to the skin friction coefficient. When \(ξ=0\), which corresponds to \(St_x \gg 1\), Eq. (7) reduces to Rayleigh type equation

\[
\frac{1}{Pr} \frac{d^2 \theta}{d \eta^2} + \frac{1}{2} Pr \frac{d \theta}{d \eta} = 0
\]  

(17)

with the boundary conditions

\[\theta(\eta=0,0) = 1 \text{ and } \theta(\eta \to \infty,0) = 0\]  

(18)

The exact solution of Eq. (34) is

\[\theta(\eta) = 1 - \text{erf} \left( \frac{Pr^{1/2}}{2} \eta \right)\]

The solution of Eq. (19) gives the dimensionless temperature profile

\[\theta(\eta,1) = 1 - \theta(0,1) \int_0^\eta \exp \left[ -\frac{1}{2} Pr \int_0^{\eta'} F(\eta'^1,1) d\eta'^1 \right] d\eta' \]  

(21)

We represent these last results in the Figures 1-2, they highlight the temperature variations according to the Prandtl number \(Pr\) for two extremes of the \(St\) values, valid in the entire interval \(\eta \in [0, \eta\infty]\).

In the convective flow, the heat performances are written as:

\[
\frac{Nu}{Re_x} = \frac{\theta(\eta,1)}{\theta(0,1)} - \left\{ \int_0^\eta \text{exp} \left[ -\frac{1}{2} Pr \int_0^{\eta'} F(\eta'^1,1) d\eta'^1 \right] d\eta' \right\}^{-1}
\]  

(22)

Which agree well with the numerical results. Table below gives the evaluation of Nusselt number for different Prandtl number values.

### Table 1. Nusselt number calculation for wide Prandtl number values in case of \(St<<1\)

<table>
<thead>
<tr>
<th>(Pr)</th>
<th>0.01</th>
<th>0.1</th>
<th>0.7</th>
<th>0.8</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Nu/Re_x^{1/2}) numerical results</td>
<td>0.051</td>
<td>0.140</td>
<td>0.292</td>
<td>0.307</td>
<td>0.332</td>
<td>0.728</td>
<td>1.571</td>
</tr>
<tr>
<td>(Nu/Re_x^{1/2}) Present approach</td>
<td>0.051</td>
<td>0.140</td>
<td>0.292</td>
<td>0.307</td>
<td>0.332</td>
<td>0.727</td>
<td>1.571</td>
</tr>
</tbody>
</table>

Figure 1. Temperature evolution \(\theta(\eta,0)\) for a wide values of \(Pr\); case of \(St<<1\).

Figure 2. Non dimensional temperature profile \(\theta(\eta,1)\) for a wide Prandtl values, \(Pr\); case of \(St>>1\).
Figure 3. Temperature profiles $\theta(\eta, St)$ distribution for different values of $St$ number for cases of $Pr = 0.01, 0.1, 0.7, 1, 7, 10$ and 100.
Figure 4. Temperature profiles $\theta(\eta, St)$ distribution for different values of $Pr$ number for cases of $St = 0.01, 0.1, 0.5, 1, 2, 10$ and 100.
It looks from Eqs. (1) and (2) the similarity between the dynamic and heat transfer boundaries layers. This property permits us to solve by the same method as previously the Eq. (8) for a wide values of the Strouhal $St$ and Prandtl $Pr$ numbers. Indeed, the proposed formula of the dimensionless temperature profile must check the boundary conditions, diffusion and steady solutions and the unsteady energy equation in the whole domain $0 \leq \eta < \eta_{\infty}$.

The interesting point of the proposed analytic function can be measured by the calculation $\Delta \varepsilon$ of equation (7)

$$
\frac{1}{P'} \frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2} \left[ \frac{\partial}{\partial \xi} \right] \left[ \frac{\partial}{\partial \eta} \right] \left[ \frac{\partial}{\partial \xi} \right] + \frac{1}{\xi} \left[ \frac{\partial}{\partial \eta} \right] \left[ \frac{\partial}{\partial \xi} \right] - \frac{1}{\xi^2} \left[ \frac{\partial^2}{\partial \xi^2} \right] \left[ \frac{\partial}{\partial \eta} \right] = \Delta \varepsilon
$$

(23)

The appropriate analytical formula of the temperature profile which converge approximately this last equation can be expressed as

$$
\theta(\eta, \xi) = 1 - \sigma \exp \left( -0.5 \frac{A(\xi)}{\eta} \right)
$$

(24)

Where $A(\xi) = \exp \left( -0.53007 \xi^{0.4} \right)$

It found that the unsteady temperature profile confirms the equation energy equilibrium, with the residual errors $\Delta \varepsilon$ does not exceed 1% for all time thus for all Strouhal numbers in all space domain. Therefore, the simple formula of $\theta(\eta, \xi)$ converges well the unsteady heat transfer boundary layer equation along a horizontal surface for all chosen values of Strouhal and Prandtl numbers.

**Evaluation of the Nusselt number**

The effectiveness of the heat transfer is measured by means of Nusselt number given by

$$
Nu_s = -R_s^{1/2} \left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0} \text{Re}^{1/2}
$$

(25)

The local heat transfer coefficient corresponding to $\xi \in [0, 1]$ and for all Strouhal numbers values $St_s$ is given by the following relation:

$$
\frac{Nu_s}{Re_s^{1/2}} = R_s^{1/2} \left( \frac{Pr^{1/2}}{\sqrt{\pi}} \exp \left( -0.53007 \xi^{0.4} \right) \right)
$$

(26)

In Figures 3-4, we represent the dimensionless heat profiles distributions corresponding to wide values of the Strouhal $St$ and Prandtl $Pr$ numbers. We note that the temperature profiles variations inside the heat boundary layer are much faster for high Prandtl numbers. This is due mainly to the viscosity effects on the heat transfer from surface to fluid flow. In addition, the temperature profiles evolutions are also very fast for Strouhal number values higher than unit $St > 1$ when we compare them with $St < 1$. We explain it by the fact that the diffusion forces due to the heat transfer between surface and fluid flow dominate the flow. Moreover, we interpret the group shift between the values of Strouhal number related to the Figure 2 by the fact that the distributions, inside the thermal boundary layer, are identical for $St_s > 100$ and for $St_s < 0.01$. From Figure 4 we note an interesting property; the temperature profiles present a quasi-identical distributions for all $St_s > 1$ and $St_s < 1$, and this at any Prandtl number values.

**RESULTS AND DISCUSSION**

The results concerning the determination of the heat transfer coefficient are well summarized on the Figures 5 and 6. We represent in Figure 5 the corresponding variations of the Nusselt number according to all Strouhal numbers, and for various Prandtl number values such as: 0.01, 0.1, 1, 10, 100. The corresponding curves reveal two linears domains of variations, in which the Nusselt number variations are more important for $St_s > 1$ and when $Pr$ increases. Furthermore, we notice a significant increase Nusselt number variations when Prandtl number increases.

Figure 5. Nusselt number evolution depending to Strouhal and Reynolds numbers for different values of Prandtl number.

Figure 6. Nusselt number evolution depending to Prandtl and Reynolds numbers for different values of Strouhal number.
Figure 6 recapitulates the main results and, since the curves are parallel, highlights the same evolution law of the Nusselt number. The increase of \( N_u \) according to \( Pr \) is sensitive of a curve to other, so when the Strouhal number \( St \) increases substantially from 1 to 10 initially then from 10 to 100.

**CONCLUSION**

In this contribution, an analytic approach is employed to study the Strouhal number effects on heat transfer evolution from Rayleigh flow to Blasius flow of a horizontal surface for various Prandtl numbers. Relevant analytic functions of the velocity and temperature profiles are proposed to approximately converging the unsteady dynamic and energy equations for all time intervals. Firstly, the realistic and powerful character related to the precision of the velocity formula is established. The obtained results made it possible to find the Rayleigh, Blasius and Williams-Rhyne solutions and to carry out the checking of the unsteady dynamic momentum equation valid for all time. Consequently, analytic expression of the skin friction coefficient is achieved for different values of Strouhal numbers. Subsequently, basing on the same reasoning as previous, an analytic temperature profile is proposed for converging the unsteady energy equation. An interesting result highlighted the impact of the flow regime on the heat transfer phenomenon near a surface were obtained, particularly, the establishment of an analytical law of the heat transfer coefficient for all Strouhal and Prandtl numbers. Depending on the Prandtl number, the Nusselt number \( N_u \) is almost constant in the case of diffusion flow \( St < 1 \) (due to the conduction transfer) while it increases linearly in both the transition regime (conduction-convection transfer) and during the convective flow \( St > 1 \) (convection transfer). In the same way, it is confirmed that, whatever the flow regime, the Nusselt number evolutions also increase when the Prandtl number increases.

**NOMENCLATURE**

**English symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( A )</td>
<td>Dimensionless function</td>
</tr>
<tr>
<td>( a )</td>
<td>Constant</td>
</tr>
<tr>
<td>( B )</td>
<td>Dimensionless function</td>
</tr>
<tr>
<td>( b )</td>
<td>Constant</td>
</tr>
<tr>
<td>( C_f )</td>
<td>Skin friction coefficient</td>
</tr>
<tr>
<td>( F )</td>
<td>Dimensionless stream function</td>
</tr>
<tr>
<td>( F_\eta )</td>
<td>Dimensionless velocity profile</td>
</tr>
<tr>
<td>( g )</td>
<td>Function depends on ( x ) and ( t )</td>
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<td>( N_u )</td>
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<tr>
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<td>Time (s)</td>
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<td>Parallel velocity to the flow (m.s(^{-1}))</td>
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<td>( v )</td>
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<td>( y )</td>
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**Greek symbols**

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<th>Symbol</th>
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<td>( \alpha )</td>
<td>Thermal diffusivity (m(^2).s(^{-1}))</td>
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<td>( \beta, \gamma )</td>
<td>Constants</td>
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<td>Convective boundary layer thickness (m)</td>
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<td>( \delta_d )</td>
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<td>( \Delta \varepsilon )</td>
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**Subscripts**

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<tbody>
<tr>
<td>( \eta )</td>
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</table>

**AUTHORSHIP CONTRIBUTIONS**

Authors equally contributed to this work.

**DATA AVAILABILITY STATEMENT**

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

**CONFLICT OF INTEREST**

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**ETHICS**

There are no ethical issues with the publication of this manuscript.

**REFERENCES**


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