ABSTRACT

The Homotopy Analysis Method (HAM) is an effective technique to achieve the analytical solution of a broad range of problems, mainly with nonlinear governing equations. The solution of Pennes’ bioheat equation in nonlinear form arising from the linear temperature-dependent nature of specific heat capacity of a biological tissue using the Homotopy Analysis Method has been obtained analytically and validated with the numerical results obtained from the Finite Difference Method (FDM) the first time in this study. The analysis demonstrated that considering the various values of the convergence parameter and computing the Mean Squared Error (MSR) to achieve the optimum values ensures accurate results even at the low-order approximations of the solution. Investigating the effect of the nonlinear term’s magnitude on the solution indicated a direct relationship; However, the effect was not remarkable even at the major values, thus it is possible to consider the specific heat capacity of a living tissue, a constant value through thermal simulations. According to this research, the Homotopy Analysis Method can be a proper method to derive the analytical solution of either the linear or nonlinear form of Pennes’ bioheat equation.

INTRODUCTION

Mathematical modeling of the heat transfer phenomenon within biological tissues has been a fascinating problem to scientists and mathematicians for over seventy years, dating back to Harry H. Pennes’ attempt to formulate bioheat transfer along the human forearm for the first time in 1948 [1]. Various models have been suggested for the heat transfer phenomenon inside living tissues since then. However, Pennes’ equation has remained the most extensively applied mathematical formulation to study the bioheat transfer phenomenon [2]. The solution of Pennes’ bioheat equation into a semi-infinite tissue was offered analytically by Shih et al. employing the Laplace transform. The heat flux followed a sinusoidal function. Shahnazari et al. found the solution of this equation for the skin exposed to a frequent
thermal flux. A procedure based on the Weighted Residual and Homotopy Perturbation Method (HPM) was applied to achieve the results [3,4]. Qin and Wu solved the Pennes' bioheat equation fractional form numerically by applying the quadratic spline collection method, whereas Al-Humedi and Al-Saadawi introduced the answer numerically using the shifted Legendre polynomials. Liu and Tu suggested the numerical results for Pennes' differential equation including transient blood temperature through the Laplace transform [5-7]. A mathematical model to present the space-fractional form of this equation was proposed by Singh et al. The solution was achieved using HPM after turning it into an IVP. Kumar et al. obtained the answer numerically utilizing the Finite Difference Method (FDM) and the Legendre wavelet Galerkin procedure. In contrast, Wang et al. developed a model for the bioheat transfer during laser irradiation based on non-Fourier heat conduction. In the following, Kabiri and Talaei solved Pennes' bioheat equation in hyperbolic form, subject to a mobile heat source analytically based on the Eigenvalue Method [8-11]. The Fibonacci wavelet method was applied to find the answer by Irfan et al. as a different approach. Time-fractional Pennes' equation solution was offered using the Fibonacci wavelet formulation by Irfan and Shah subsequently [12,13].

Lakhssassi et al. proposed a simplified formulation according to the modified format of this bioheat equation at the steady state. The solutions were found using the Weierstrass elliptic function. Gupta et al. presented a model for heat transfer within thermal therapy using electromagnetic radiation. The solution was achieved by the Galerkin method with the B-polynomial, and the problem was solved by HPM [14,15]. Majchrzak offered the solution by the Boundary Element Method (BEM) for a dual-phase-lag problem. Majchrzak and Turchan also obtained a three-dimensional solution to a similar problem by general BEM [16,17].

Numerous attempts have been made thus far in the pursuit of finding numerical and analytical solutions for nonlinear heat transfer equations [18-21]. Karimipour et al. conducted a study to investigate the effects of gravity on mixed convection heat transfer in a microchannel using the Lattice Boltzmann Method [22]. In similar research, the effects of magnetic field and slip were studied on the movement of a nanofluid inside a magnetic field by OHAM, and Obalalu et al. studied a Casson fluid flow applying HAM [38-40]. Rabbani introduced a modified HPM to obtain the answer for nonlinear integral equations. Jbr and Al-Rammahi used q-HAM to solve the Fredholm integral equation in nonlinear form. Abdulkhaleq applied a combination of HAM and the Harris Hawks method to solve several PDEs [41-43].

In this study, Pennes' bioheat equation with a nonlinear term arising from the linear temperature-dependent nature of the specific heat capacity of biological tissue has been solved using HAM for the first time. The accuracy was evaluated by comparing the results with the numerical answers derived from the Finite Difference Method (FDM) and calculating the Mean Squared Error.

**PENNES’ BIOHEAT EQUATION**

As it was mentioned in the introduction section, Pennes’ bioheat equation is the most common mathematical formulation for studying bioheat transfer. The Pennes’ bioheat equation in general is shown in the form [44]:

\[
\rho c \frac{dT}{dt} = \nabla \cdot (k \nabla T) - \rho_b c_b \omega_p (T - T_s) + q_m
\]  

(1)

By assuming a fixed thermal conductivity and blood perfusion rate for the biological tissue \((k, \omega_p = cte)\) the one-dimensional form of (1) is derived as:

\[
\rho c \frac{dT(x,t)}{dt} = k \frac{\partial^2 T(x,t)}{\partial x^2} - \rho_b c_b \omega_p (T(x,t) - T_s) + q_m ; \quad 0 < x < L
\]  

(2)

which \(x\) denotes the spatial variable of this equation. The initial condition of the equation is considered as:
Defining the dimensionless groups below will result in the nondimensional form of (2).

\[X = \frac{x}{L}; \quad \tau = \frac{t}{\omega_c}; \quad \theta = \frac{T}{T_a}; \quad \alpha_1 = \frac{k}{\rho_c \omega_c L^2};\]
\[\alpha_2 = \frac{\rho_l c_h}{\rho_c}; \quad \alpha_3 = \frac{1}{\rho_c} (\rho_l c_h + \frac{q_u}{\omega_c T_a})\]

(4)

The nondimensional form of (2) regarding the introduced dimensionless groups is:

\[\frac{\partial \theta(X, \tau)}{\partial \tau} = \alpha_1 \frac{\partial^2 \theta(X, \tau)}{\partial X^2} - \alpha_2 \theta(X, \tau) + \alpha_3; \quad 0 \leq X \leq 1\]

(5)

and the nondimensional initial condition is expressed as:

\[\theta(X, 0) = 1\]

(6)

Considering the tissue’s specific heat capacity as the temperature’s linear function, namely [45]:

\[c_i(T(x, t)) = c_0 (1 + \beta(T(x,t) - T_a))\]

(7)

and substituting it into (2) with the consideration of (8) results in (9). Here \(c_0\) denotes the tissue’s specific heat capacity at core body temperature, and \(\beta(\Theta^{-1})\) is a constant.

\[\varepsilon = \beta T_a\]

(8)

\[\frac{(1 + \varepsilon(\theta(X, \tau) - 1))}{\partial \tau} = \alpha_1 \frac{\partial^2 \theta(X, \tau)}{\partial X^2} - \alpha_2 \theta(X, \tau) + \alpha_3; \quad 0 \leq X \leq 1\]

(9)

Homotopy Analysis Method (HAM)

Let us consider nonlinear Pennes’ bioheat equation as follows [46]:

\[N[\theta(X, \tau)] = 0\]

(10)

which \(N\) denotes the nonlinear operator, \(X\) and \(\tau\) are the dimensionless spatial and temporal variables, and \(\theta(X, \tau)\) is the unknown solution respectively. Assuming \(\theta_0(X, \tau)\) as the initial guess, \(h \neq 0\) the convergence-control parameter, \(H(X, \tau) \neq 0\) the auxiliary function, and \(L\) the linear auxiliary operator with the character:

\[L(f) = 0 \quad \text{when} \quad f = 0\]

(11)

and assuming \(q \in [0,1]\), a homotopy is constructed as below:

\[(1-q)L[\Phi(X, \tau; q) - \theta_0(X, \tau)] = q H(X, \tau) N[\Phi(X, \tau; q)]\]

(12)

When \(q = 0\):

\[L[\Phi(X, \tau; 0) - \theta_0(X, \tau)] = 0\]

(13)

Referring to (11):

\[\Phi(X, \tau; 0) = \theta_0(X, \tau)\]

(14)

When \(q = 1\):

\[H(X, \tau) N[\Phi(X, \tau; 1)] = 0\]

(15)

Since \(h, H(X, \tau)\) are nonzero items, consequently:

\[N[\Phi(X, \tau; 1)] = 0\]

(16)

and this means:

\[\Phi(X, \tau; 1) = \theta(X, \tau)\]

(17)

As it was demonstrated above, when \(q\) grows from 0 to 1, \(\Phi(X, \tau; q)\) continuously converts from the initial guess into the analytical solution of nonlinear Pennes’ bioheat equation which was introduced in (9). Expanding \(\Phi(X, \tau; q)\) into the Taylor series respecting \(q\) results in the following equation:

\[\Phi(X, \tau; q) = \Phi(X, \tau; 0) + \sum_{n=1}^{\infty} \frac{1}{m!} \frac{\partial^n \Phi(X, \tau; q)}{\partial q^n} \bigg|_{q=0} q^n\]

(18)

By defining the m-th order deformation derivative in the form:

\[\theta_m(X, \tau) = \frac{1}{m!} \frac{\partial^n \Phi(X, \tau; q)}{\partial q^n} \bigg|_{q=0}\]

(19)

\(\Phi(X, \tau; q)\) is expanded in the following form, according to the Taylor expansion of \(\Phi(X, \tau; q)\), which was shown in (18):

\[\Phi(X, \tau; q) = \theta_q(X, \tau) + \sum_{n=1}^{\infty} \theta_n(X, \tau) q^n\]

(20)

Regarding (17) by setting \(q = 1\) in (20), the solution of nonlinear Pennes’ bioheat equation emerges as:

\[\theta(X, \tau) = \theta_q(X, \tau) + \sum_{n=1}^{\infty} \theta_n(X, \tau)\]

(21)

The following vector is considered:

\[\overrightarrow{\theta} = \{\theta_q(X, \tau), \theta_1(X, \tau), \theta_2(X, \tau), ..., \theta_n(X, \tau)\}\]

(22)
To determine $\theta_m(X, \tau)$, the m-th order derivative of the zero-order deformation (12) with respect to $q$ is computed, then the result is divided by $m!$ and finally $q$ is set to 0. Through following this process, the high-order deformation appears in the form:

$$L[\theta_m(X, \tau) - \chi_m \theta_m(X, \tau)] = h H(X, \tau) R_m(\theta_{m-1}, X, \tau)$$ (23)

and $\theta_m(X, \tau)$ is calculated as:

$$\theta_m(X, \tau) = \chi_m \theta_{m-1}(X, \tau) + h L^{-1}[H(X, \tau) R_m(\theta_{m-1}, X, \tau)]$$ (24)

By introducing $\chi_m$ in the following form:

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}$$ (25)

and $R_m(\theta_{m-1}, X, \tau)$ in the form:

$$R_m(\theta_{m-1}, X, \tau) = \frac{1}{(m-1)!} \frac{\partial^{m-1}(\Phi(X, \tau, q))}{\partial q^{m-1}} |_{q=0}$$ (26)

besides substituting (25) and (26) into (24), $\theta(X, \tau)$ is calculated by (21). The m-th order approximation of $\theta(X, \tau)$ is presented as:

$$\theta(X, \tau) \approx \sum_{i=0}^{m} \theta_i(X, \tau)$$ (27)

The Analytical Solution Built on Homotopy Analysis Method

The analytical solution of (9) obtained by HAM has been investigated in this section. The boundary conditions in this article are defined as:

$$T(x, t)|_{x=0} = T_a \rightarrow \theta(X, \tau)|_{X=0} = 1$$ (28)

$$T(x, t)|_{x=L} = 0 \rightarrow \theta(X, \tau)|_{X=L} = 0.$$ (29)

The linear operator is chosen $L = \frac{\partial}{\partial \tau}$, the nonlinear one is selected in the form $N = \frac{1}{1+\varepsilon} \left[ \frac{\partial^2}{\partial X^2} - \alpha \frac{\partial^2}{\partial \tau^2} + \alpha_{12} + \alpha_{13} \right]$ and $H(X, \tau) = 1$ for simplification purposes here.

The initial guess is selected according to the boundary conditions (28), (29) as well as the right side of (9).

$$\frac{\partial}{\partial \tau} (\theta_b(X, \tau)) = 0 \rightarrow \theta_b(X) = -\frac{2 \sinh(X)}{e^{-e^{-1}}} + 1$$ (30)

The m-th order approximation of the solution can be calculated referring to (24) with the boundary conditions below:

$$\theta_m(X, \tau)|_{X=0} = 0 \text{ and } \theta_m(X, \tau)|_{X=L} = 0.$$ (31)

A Maple code was written to implement the elaborated procedure and obtain the Pennes’ nonlinear bioheat equation’s analytical solution using HAM in this study.

RESULTS AND DISCUSSION

Assuming $\varepsilon = 1$, $\alpha_i = 1$ ($i = 1, 2, 3$), and following the described process will result in the m-th order approximate solution of (9) with the relevant boundary conditions. The zero and the first-order approximations are in the form:

$$\theta_0(X, \tau) = \frac{-e^{X+1} + e^{X+1} + e^2 - 1}{e^2 - 1}$$ (32)

$$\theta_1(X, \tau) = \frac{h \tau - e^{X-1} - e^{X+1} + e^2 - 1}{e^2 - 1}$$ (33)

Repeating this algorithm will result in higher-order deformations and more accurate approximations. It is evident that the first-order analytical solution and the numerical one achieved by the Finite Difference Method (FDM) converge at $h = 1.0$ with a remarkable level of accuracy as illustrated through Figure 1.

It is clear that the answer of (5) can be obtained by setting $\varepsilon = 0$. For the high computational cost of HAM in finding the higher-order approximate solutions of (9), the effect of the convergence parameter has only been studied on the

![Figure 1](image-url)
first and second-order approximations at \( \alpha_i = 1 \) \( (i = 1, 2, 3) \), \( \epsilon = 1 \), \( \tau = 1 \). The results have been noted in Tables 1 and 2. The Mean Squared Error (MSE) has been calculated in the following form for each \( h \) values [47]:

\[
MSE = \frac{1}{4} \sum_{i=1}^{4} [\theta_{\text{analytical}} (X_i) - \theta_{\text{numerical}} (X_i)]^2
\]  

(34)

MSE for the first and second-order approximations of the solution regarding the different \( h \) values has been depicted in Figure 2. It is easily recognized that with the increase of \( h \), the error decreases gradually, and after a specific value, the error rises again. As can be observed, the optimum \( h \) values to achieve the least MSE for the first and second-order approximations, are 1.0 and 0.6 in order for the studied case. It demonstrates that by selecting an appropriate value for the convergence parameter, achieving accurate results even at low-order approximations is possible, and it is the superiority of HAM in comparison to other power-series-based analytical techniques.

The effect of \( \epsilon \) on the second-order approximate solution with \( h = 1.0 \), \( \tau = 1 \) has been investigated and shown in Figure. 3. Since the small values of \( \epsilon \) do not affect the solution remarkably, the results for the minor values have been indicated in Table 3. As can be recognized through the graph and table, the results grow with the increase in \( \epsilon \). The second-order approximate solutions of nonlinear Pennes’ bioheat as a function of \( X \), \( \tau \) at \( \epsilon = 0, 20, 50, 100, \alpha_i = 1 \) \( (i = 1, 2, 3) \), \( h = 1.0 \) for \( 0 \leq X \leq 1 \) and \( 0 \leq \tau \leq 1 \) have been displayed in Figure 4.

<table>
<thead>
<tr>
<th>Table 1. MSE for the first-order approximate solution of (9) using HAM at the various ( h ) values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
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<tr>
<td>------</td>
</tr>
<tr>
<td>0.1</td>
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<tr>
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<td>0.7</td>
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<td>0.8</td>
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<tr>
<td>0.9</td>
</tr>
<tr>
<td>1.0</td>
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<tr>
<td>1.1</td>
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<tr>
<td>Finite Difference Solution</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Table 2. MSE for the second-order approximate solution of (9) using HAM at the various ( h ) values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
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<td>------</td>
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<tr>
<td>0.1</td>
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<td>0.3</td>
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<td>0.7</td>
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<td>0.8</td>
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<tr>
<td>0.9</td>
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<tr>
<td>1.0</td>
</tr>
<tr>
<td>1.1</td>
</tr>
<tr>
<td>Finite Difference Solution</td>
</tr>
</tbody>
</table>
Figure 2. MSE for the first and second-order approximations according to the $h$ values.

Table 3. The solution of (9) at $h = 1.0$, $\tau = 1$ with the minor values of $\epsilon$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$X = 0.2$</th>
<th>$X = 0.4$</th>
<th>$X = 0.6$</th>
<th>$X = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon = 0.0$</td>
<td>0.931757074</td>
<td>0.803379825</td>
<td>0.61115635</td>
<td>0.34732048</td>
</tr>
<tr>
<td>$\epsilon = 0.1$</td>
<td>0.931994893</td>
<td>0.803810506</td>
<td>0.61164614</td>
<td>0.347712304</td>
</tr>
<tr>
<td>$\epsilon = 0.2$</td>
<td>0.932232706</td>
<td>0.804241189</td>
<td>0.61213593</td>
<td>0.348052554</td>
</tr>
<tr>
<td>$\epsilon = 0.5$</td>
<td>0.932946156</td>
<td>0.805533236</td>
<td>0.61360531</td>
<td>0.349073312</td>
</tr>
<tr>
<td>$\epsilon = 1.0$</td>
<td>0.934135235</td>
<td>0.807686645</td>
<td>0.61605426</td>
<td>0.350774575</td>
</tr>
</tbody>
</table>

Figure 3. The second-order approximate solutions of (9) at $h = 1.0$, $\alpha_i = 1$ ($i = 1,2,3$), $\tau = 1$ with various $\epsilon$ values.
CONCLUSION

The comparison between the analytical answer of Pennes' equation arising from the Homotopy Analysis Method and the numerical solution demonstrates a very good agreement. It is not easy to obtain the high-order deformations of nonlinear Pennes' bioheat equation for the high computational cost of HAM; However, achieving the acceptable accuracy by applying the convergence-control parameter's optimum values even at the low-order approximations is possible. Analyzing the effect of \( \varepsilon \) on the solution indicates that the increase in this parameter's value will increase the temperature. Nevertheless, the effect is incon siderable even for the large values of \( \varepsilon \), thus it is reasonable to consider the specific heat capacity of a living tissue independent of its temperature (a constant value). Referring to this study, HAM is considered an appropriate analytical technique to compute the solution of Pennes' equation in a nonlinear form.

NOMENCLATURE

- \( T \) Temperature
- \( t \) Time
- \( \rho_t \) Density of tissue
- \( c_t \) Specific heat capacity
- \( k \) Thermal conductivity
- \( \rho_b \) Density of blood
- \( c_b \) Specific heat capacity of blood
- \( \omega_p \) Blood perfusion rate
- \( T_a \) Core body temperature
- \( q_{im} \) Volumetric metabolic heat generation

Figure 4. The second-order approximate solutions of nonlinear Pennes' bioheat equation at \( \varepsilon = 0, 20, 50, 100, \alpha_i = 1.0 \) (\( i = 1,2,3 \)), \( h = 1.0 \).
AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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