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Coupled Flexural-Lateral-Torsional of Shear Deformable Thin-Walled Beams with Asymmetric Cross-Section – Closed Form Exact Solution

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ABSTRACT

This paper develops the exact solutions for coupled flexural-lateral-torsional static response of thin-walled asymmetric open members subjected to general loading. Using the principle of stationary total potential energy, the governing differential equations of equilibrium are formulated as well as the associated boundary conditions. The formulation is based on a generalized Timoshenko-Vlasov beam theory and accounts for the effects of shear deformation due to bending and warping, and captures the effects of flexural–torsional coupling due to cross-section asymmetry. Closed-form solutions are developed for cantilever and simply supported beams under various forces. In order to demonstrate the validity and the accuracy of this solution, numerical examples are presented and compared with well-established ABAQUS finite element solutions and other numerical results available in the literature. In addition, the results are compared against non-shear deformable beam theories in order to demonstrate the shear deformation effects.

LITERATURE REVIEW ON ANALYTICAL SOLUTIONS

Given the vast literature on the subject, the present literature survey focuses on the static and dynamic analysis of thin-walled shear deformable beams with asymmetric open cross-sections, i.e., members with doubly symmetric and monosymmetric cross-sections are not included in the present survey. Among them, Bercin and Tanaka [1] studied the coupled flexural-torsional free vibrations of thin-walled members of asymmetric open C-sections. Kim et. al [2] formulated the exact dynamic and static stiffness matrices for the free vibration and stability analysis of thin-walled shear-deformable beams. Also, they incorporated flexural-torsional coupling effects due to the asymmetry of the cross-sections. In a subsequent study, Kim and Kim [3] adopted the theory in [2] to formulate the dynamic stiffness matrix element for the flexural-torsional free vibration of asymmetric shear-deformable thin-walled beams. Li et. al [4] developed the dynamic transfer matrix to formulate a solution for determining the coupled bending-torsional response of thin-walled beam under random excitations by considering the effects of warping stiffness and rotary inertia. Prokic [5] formulated the governing equations for the coupled bending-torsional vibrations of thin-walled beams. Vo and Lee [6] presented a general analytical solution for the study of flexural-torsional buckling and vibration analysis of open thin-walled composite beams. Jung and Lee [7] derived a closed-form solution for both symmetric and antisymmetric lay-up I-beam. Their solution included the effects of torsional warping and constrained warping. Kim et al. [8] derived the exact stiffness matrices for the buckling and the elastic analysis of thin-walled beam with nonsymmetric cross-sections. Ambrosini [9] developed a general theory for coupled flexural-torsional free vibrations for thin-walled beams of open cross-sections. De Bordon [10] extended the theory for coupled flexure and torsion vibrations of thin-walled beams to incorporating the influence of the axial forces. In Ambrosini [11], an experimental study for the free vibration of thin-walled beams with asymmetric open cross section was conducted and the results were used to assess the accuracy of various theoretical solutions. The above studies accounts for the effects of shear deformation, warping and rotary inertia.

Although a large number of studies have been developed to investigate the static and dynamic response of thin-walled open asymmetric beams, to the best of the author's knowledge, no closed-form solutions have been reported for the transverse-lateral-torsional-warping coupled static response of thin-walled asymmetric beams which account for shear deformation effects due to bending and warping. Within the above context, the present study aims at developing exact closed-form solution for static response of shear deformable thin-walled open beams with asymmetric cross-sections.

KINEMATICS RELATIONS

A thin-walled member of arbitrary open cross-section has a fixed right-handed orthogonal Cartesian coordinate system (X,Y,Z) with the Z axis parallel to the longitudinal axis of the beam used to describe the geometry and displacements. Fig. 1 shows a local coordinate system (n,s,z) positioned on the contour (middle line of the cross-section) in which the coordinates n and s are measured along the normal and along the tangent to the middle surface at the contour point of interest.

The present theory of thin-walled asymmetric cross-section is based on the following assumptions:

1. The formulation is applicable to prismatic thin-walled members of arbitrary open cross-sections,
2. Cross-section is assumed to remain undeformed in its own plane (the first Vlasov assumption), but free to warp in the longitudinal direction,
3. Under loading not involving twisting effects, the cross-section remains planar but does not remain perpendicular to the centroidal axis after deformation, i.e., the transverse shear deformation of the mid-surface of the cross-section is incorporated in the assumed kinematics (Timoshenko beam assumption),
4. Under twisting effects, the section is assumed to undergo warping characterized by the Vlasov warping function [12], although the second Vlasov assumption which assumes zero shear strains within the middle surface (i.e., the second Vlasov assumption) is relaxed,
5. The material behavior is assumed to remain linearly elastic throughout deformation,
6. Strains and rotations are assumed small.

Based on the above assumptions, the in-plane displacements $u_p(z,s)$, $v_p(z,s)$ and longitudinal displacement $w_p(z,s)$ of a general point $p(x(s),y(s))$ located on the mid-surface of the cross-section are respectively given by:

$$v_p(z,s)=v(z)+\left[x(s)-x_s\right]\theta_z(z) \tag{1}$$

$$u_p(z,s)=u(z)-\left[y(s)-y_s\right]\theta_z(z) \tag{2}$$

$$w_p(z,s)=w(z)+y(s)\theta_x(z)-x(s)\theta_y(z)+\omega(s)\psi(z) \tag{3}$$

where $u(z)$ and $v(z)$ are the displacements of the shear centre S_c along the principal directions (X,Y), $w(z)$ is the average longitudinal displacement along the longitudinal axis z, $\theta_x(z)$ and $\theta_y(z)$ are the rotations of the cross-section about X and Y principal axes, respectively, $\theta_z(z)$ is the rotation angle of the cross-section about the longitudinal axis, $\psi(z)$ is a function which characterizes the magnitude of the warping deformation, $\omega(s)$ is the Vlasov warping function defined by $\omega(s)=\int_A h(s)\,dA$, $x(s)$ and $y(s)$ are the coordinates of a point denoted by a curvilinear coordinate s lying on the middle surface of the section, while x_s and y_s are the coordinates of the shear centre along the principal axes.

The in-plane displacement functions $u_p(z,s)$ and $v_p(z,s)$ of a general point p are resolved into tangential displacement $\xi(z,s)$ along local coordinates s and n, (Fig. 1), yielding:

$$\xi(z,s)=u(z)cos\alpha+v(z)\sin\alpha+h(s)\theta_z(z) \tag{4}$$

where $h(s)=\left[x(s)-x_s\right]\sin\alpha-\left[y(s)-y_s\right]\cos\alpha$, $\sin\alpha=dy(s)/ds$, $\cos\alpha=dx(s)/ds$, $\alpha(s)$ is the angle between the tangent of the cross-section of point p and the X axis, and $h(s)$ is the distance from the shear center perpendicular to the tangent to the contour at point p (Fig. 1).

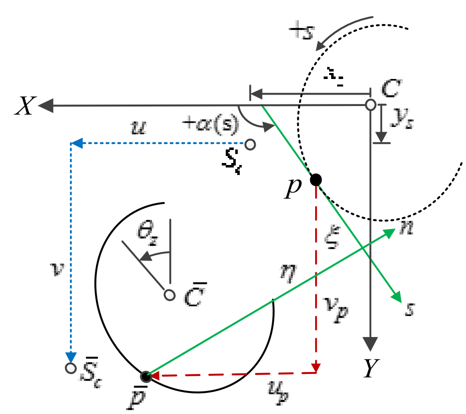


FIGURE 1 COORDINATE SYSTEMS AND DISPLACEMENTS

VARIATIONAL FORMULATION

The total potential energy Π of the thin-walled beam is defined as the sum of the internal strain energy U stored in the deformed body and the potential energy V due to applied loads, i.e., $\Pi = U + V$. Taking the first variation of Π and setting it equal to zero, one obtains:

$$\delta\Pi = \delta U + \delta V = 0 \tag{5}$$

in which δU is the internal strain energy given by:

$$\delta U = \int_0^{\ell}\int_A \left[E\varepsilon_{zz}\delta\varepsilon_{zz} + G\gamma_{zs}\delta\gamma_{zs}\right] dA dz + \int_0^{\ell} GJ\theta_z'\delta\theta_z' dz \tag{6}$$

where E is the elasticity modulus, G is the shear modulus, J is the Saint Venant torsional constant, A is the cross-sectional area, $\varepsilon_{zz} = \partial w_p / \partial z$ and $\gamma_{zs} = \partial w_p / \partial s + \partial \xi / \partial z$ are the longitudinal and shear strains, respectively, and all primes denote derivatives with respect to coordinate z . The potential of the applied forces δV is given by:

$$\begin{aligned} \delta V = \int_0^{\ell} \Big[& q_z\delta w + q_x\delta u + q_y\delta v + m_z\delta\theta_z + m_x\delta\theta_x + m_y\delta\theta_y \\ & + m_w\delta\psi\Big] dz + \left[N_z\delta w\right]_0^{\ell} + \left[V_x\delta u\right]_0^{\ell} + \left[V_y\delta v\right]_0^{\ell} \\ & + \left[M_x\delta\theta_x\right]_0^{\ell} + \left[M_y\delta\theta_y\right]_0^{\ell} + \left[M_z\delta\theta_z\right]_0^{\ell} + \left[M_w\delta\psi\right]_0^{\ell} \end{aligned} \tag{7}$$

where $q_z(z), q_x(z), q_y(z)$ are the distributed longitudinal, transverse and lateral forces, $m_x(z), m_y(z)$ and $m_z(z)$ are the distributed bending and twisting moments, $N_z(z_e)$, $V_x(z_e)$, $V_y(z_e)$ are the concentrated longitudinal, transverse and lateral forces, $M_z(z_e)$, $M_x(z_e)$, $M_y(z_e)$ are the end moments and $M_w(z_e)$ is the end bimoment, and all forces and moments applied at beam ends $(z_e = 0, \ell)$. All applied forces are assumed to have the same sign convention as those of the end displacements (Fig. 1).

EQUILIBRIUM GOVERNING FIELD EQUATIONS

From equations (1-4), by substituting into equations (6-7). The resulting energy equations are substituted into equation (5) and by enforcing the orthogonality conditions; $\left\langle \int_A \left[x, y, xy, x\omega, y\omega, \omega\right] dA \right\rangle = \langle 0 \rangle$ and performing integration by parts with respect to coordinate z, the governing equilibrium equations are then obtained as:

$$EAw'' = -q_z(z) \tag{8}$$

$$G\left[D_{xx}\left(u'' - \theta_y'\right) + D_{xy}\left(v'' + \theta_x'\right) + D_{hx}\left(\theta_z'' + \psi'\right)\right] = -q_x(z) \tag{9}$$

$$G\left[D\left(u'' - \theta'\right) + D\left(v'' + \theta'\right) + D\left(\theta'' + \psi'\right)\right] = -q\left(z\right) \tag{10}$$

$$\begin{aligned} -EI_{xx}\theta_x'' + G\Big[& D_{xy}\left(u' - \theta_y\right) + D_{yy}\left(v' + \theta_x\right) \\ & + D_{hy}\left(\theta_z' + \psi\right)\Big] = m_x(z) \end{aligned} \tag{11}$$

$$\begin{aligned} -EI_{yy}\theta_y'' - G\Big[& D_{xx}\left(u' - \theta_y\right) + D_{xy}\left(v' + \theta_x\right) \\ & + D_{hx}\left(\theta_z' + \psi\right)\Big] = m_y(z) \end{aligned} \tag{12}$$

$$\begin{aligned} GJ\theta_z'' + G\Big[& D_{hx}\left(u'' - \theta_y'\right) + D_{hy}\left(v'' + \theta_x'\right) \\ & + D_{\omega\omega}\left(\theta_z'' + \psi'\right)\Big] = -m_z(z) \end{aligned} \tag{13}$$

$$\begin{aligned} EC_w\psi'' - G\Big[& D_{hx}\left(u' - \theta_y\right) + D_{hy}\left(v' + \theta_x\right) \\ & + D_{\omega\omega}\left(\theta_z' + \psi\right)\Big] = m_w(z) \end{aligned} \tag{14}$$

The related boundary conditions are obtained as:

$$\left[EAw' - N_z\right]\delta w(z)\Big|_0^{\ell} = 0 \tag{15}$$

$$\left[G\left\{D_{xx}\left(u' - \theta_y\right) + D_{xy}\left(v' + \theta_x\right) + D_{hx}\left(\theta_z' + \psi\right)\right\} - V_x\right]\delta u\Big|_0^{\ell} = 0 \tag{16}$$

$$\left[G\left\{D\left(u' - \theta\right) + D\left(v' + \theta\right) + D\left(\theta' + \psi\right)\right\} - V\right]\delta v(z)\Big| = 0 \tag{17}$$

$$\left[EI_{xx}\theta_x' + M_x\right]\delta\theta_x(z)\Big|_0^{\ell} = 0 \tag{18}$$

$$\left[EI_{yy}\theta_y' - M_y\right]\delta\theta_y(z)\Big|_0^{\ell} = 0 \tag{19}$$

$$\begin{aligned} \Big[& GJ\theta_z' + GD_{hx}\left(u' - \theta_y\right) + GD_{hy}\left(v' + \theta_x\right) \\ & + GD_{\omega\omega}\left(\theta_z' + \psi\right) - M_z\Big]\delta\theta_z(z)\Big|_0^{\ell} = 0 \end{aligned} \tag{20}$$

$$\left[EC_w\psi' + M_w\right]\delta\psi(z)\Big|_0^{\ell} = 0 \tag{21}$$

In the above equations, the following cross-sectional properties have been defined by:

$$A, I_{xx}, I_{yy} = \int_A \left[1, y^2, x^2\right] dA \tag{22a}$$

$$\begin{aligned} D_{xx}, D_{yy}, D_{xy}, D_{hx}, D_{hy}, D_{\omega\omega} = \int_A \Bigg[& \left(\frac{dx}{ds}\right)^2, \left(\frac{dy}{ds}\right)^2, \\ & \left(\frac{dy}{ds}\right)\left(\frac{dy}{ds}\right), \left(\frac{d\omega}{ds}\right)\left(\frac{dx}{ds}\right), \left(\frac{d\omega}{ds}\right)\left(\frac{dy}{ds}\right), \left(\frac{d\omega}{ds}\right)^2 \Bigg] dA \end{aligned} \tag{22b}$$

Equation (8) governs the static longitudinal response of the beam and is uncoupled from the remaining field equations and can be solved. In contrast, equations (9-14) and associated boundary conditions (15-21) govern the coupled biaxial bending-torsional-warping static response. Unlike the governing equations of the Vlasov theory which happen to be uncoupled, the present shear deformable theory happens to lead to fully coupled field equations. The present work focuses only on the analytical closed-form solution for the coupled system of equations (9-14).

HOMOGENEOUS SOLUTION FOR COUPLED BENDING-TORSIONAL EQUATIONS

The homogeneous solution of the governing coupled bending-torsional equations (9) to (14) is obtained by setting the loading terms in the coupled field equations to zero, i.e., $q_x(z)=q_y(z)=m_x(z)=m_y(z)=m_z(z)=m_w(z)=0$.

The solution of the displacement functions is assumed to take the following exponential form:

$$\begin{aligned}\left\langle \overline{W}(z)\right\rangle_{1\times 6} &= \left\langle u_h(z) \,\vdots\, v_h(z) \,\vdots\, \theta_{x_h}(z) \,\vdots\, \theta_{y_h}(z) \,\vdots\, \theta_{z_h}(z) \,\vdots\, \psi_h(z)\right\rangle \\ &= \left\langle c_i\right\rangle_{1\times 6} e^{m_i z}, \quad \text{for } i=1,2,3,\ldots,6\end{aligned} \tag{23}$$

in which $\left\langle \overline{W}(z)\right\rangle_{1\times 6} = \left\langle u_h(z) \,\vdots\, v_h(z) \,\vdots\, \theta_{x_h}(z) \,\vdots\, \theta_{y_h}(z) \,\vdots\, \theta_{z_h}(z) \,\vdots\, \psi_h(z)\right\rangle_{1\times 6}$ is the vector of transverse, lateral, torsional and warping deformation functions, and $\left\langle c_i\right\rangle_{1\times 6} = \left\langle c_1 \; c_2 \; c_3 \; c_4 \; c_5 \; c_6\right\rangle_{1\times 6}$ is the vector of unknown integration constants. From the displacement functions in equation (23), by substituting into coupled equations (8-14), rewritten in matrix form, one obtains:

$$\begin{bmatrix} S_{11}m_i^2 & S_{12}m_i^2 & S_{13}m_i & S_{14}m_i & S_{15}m_i^2 & S_{16}m_i \\ & S_{22}m_i^2 & S_{23}m_i & S_{24}m_i & S_{25}m_i^2 & S_{26}m_i \\ & & S_{22}-S_{33}m_i^2 & S_{34} & S_{35}m_i & S_{36} \\ & & & S_{11}-S_{44}m_i^2 & S_{45}m_i & S_{46} \\ & Symm & & & \left(S_{56}+S_{55}\right)m_i^2 & S_{56}m_i \\ & & & & & S_{56}-S_{66}m_i^2 \end{bmatrix}_i \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}_i = \{0\} \tag{24}$$

in which $S_{11}=GD_{xx}$, $S_{12}=S_{13}=GD_{xy}$, $S_{14}=-S_{11}$, $S_{15}=S_{16}=GD_{hx}$, $S_{22}=S_{23}=GD_{yy}$, $S_{24}=-S_{12}=S_{34}$, $S_{25}=S_{26}=S_{35}=GD_{hy}$, $S_{33}=EI_{xx}$, $S_{44}=EI_{yy}$, $S_{45}=S_{46}=-S_{15}$, $S_{55}=GJ$, $S_{56}=GD_{\omega\omega}$, $S_{66}=EC_w$ and $S_{ij}=S_{ji}\;\; if \;\; i\neq j$, where $r_o^2 = x_s^2 + y_s^2 + \left(I_{xx}+I_{yy}\right)/A$ is the polar radius of gyration about the shear centre. For a non-trivial solution, the determinant of the matrix in equation (24) is set to vanish leading to the quadratic eigenvalue problem of the form:

$$\left(m_i^2\left[\widehat{M}\right]_{6\times 6} + m_i\left[\widehat{C}\right]_{6\times 6} + \left[\widehat{K}\right]_{6\times 6}\right)\{c\}_{i,6\times 1} = 0 \tag{25}$$

where $\{c\}_i$ are the eigenvectors corresponding to eigenvalues m_i, matrices $\left[\widehat{M}\right]_{6\times 6}$, $\left[\widehat{C}\right]_{6\times 6}$ and $\left[\widehat{K}\right]_{6\times 6}$ are defined by

$$\left[\widehat{M}\right]_{6\times 6} = \begin{bmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ & S_{22} & 0 & 0 & S_{25} & 0 \\ & & -S_{33} & 0 & 0 & 0 \\ & & & -S_{44} & 0 & 0 \\ & Symm & & & S_{55}+S_{56} & 0 \\ & & & & & -S_{66} \end{bmatrix}_{6\times 6},$$

$$\left[\widehat{C}\right]_{6\times 6} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} & 0 & S_{16} \\ & 0 & S_{23} & S_{24} & 0 & S_{26} \\ & & 0 & 0 & S_{35} & 0 \\ & & & 0 & S_{45} & 0 \\ & Sym. & & & 0 & S_{56} \\ & & & & & 0 \end{bmatrix}_{6\times 6}, \text{ and}$$

$$\left[\hat{K}\right]_{6\times6} = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 \\ \hline & & S_{22} & S_{34} & 0 & S_{36} \\ \hline & & & S_{11} & 0 & S_{46} \\ \hline & Sym. & & & 0 & 0 \\ \hline & & & & & S_{56} \end{array}\right]_{6\times6}.$$

The quadratic 6×6 eigenvalue problem defined in equation (24) is transformed into an equivalent 12×12 unsymmetrical linear eigenvalue problem as:

$$\left(m_i\left[\begin{array}{c|c}[I_6] & [0] \\ \hline [0] & [\hat{M}]\end{array}\right]-\left[\begin{array}{c|c}[0] & [I_6] \\ \hline [-\bar{K}] & [-\bar{C}]\end{array}\right]\right)\left\{\begin{array}{c}\{c\}_i \\ \hline m_i\{c\}_i\end{array}\right\}_{i,12\times12}=\{0\} \quad (26)$$

where $[I_6]$ is the 6×6 identity matrix. The non-trivial solution of equation (26) is given by the right eigen-value arising by setting the determinant of the matrix in equation (26) to vanish. The generalized eigenvalues and corresponding eigenvectors are then determined numerically. For thin-walled beams with asymmetric cross-section, it is observed that all twelve roots are non-zero and distinct (i.e., $m_i \neq m_j$ for $i \neq j$). Thus, the homogeneous solution of system of coupled equations (8-14) takes the form:

$$\{\bar{W}(z)\}_{6\times1}=[\bar{G}]_{6\times12}[\bar{E}(z)]_{12\times12}\{C_i\}_{12\times1} \quad (27)$$

where $[\bar{E}(z)]_{12\times12}=Diag\left[e^{m_1z}\;|\;e^{m_2z}\;|\;e^{m_3z}\;|\;....\;|\;e^{m_{12}z}\right]_{12\times12}$, is a diagonal matrix of exponential functions, $[\bar{G}]_{6\times12}$ is the matrix of eigen vectors, and $\{C_i\}_{12\times1}$ is a vector of unknown constants to be determined from the boundary conditions.

Solution for Cantilever Beam under End Forces

For a cantilever with asymmetric section under end forces; transverse force $P_y(\ell)$, lateral force $P_x(\ell)$, bending moments $M_x(\ell)$, $M_y(\ell)$, twisting moment $M_z(\ell)$ and bimoment $M_w(\ell)$, the unknown constants $\{C_i\}_{12\times1}$ are determined from the boundary conditions at both ends, $z=0,\ell$:

$$u(0)=v(0)=\theta_x(0)=\theta_y(0)=\theta_z(0)=\psi(0)=0 \quad (28\text{-}33)$$

$$GD_{xx}\left[u'(\ell)-\theta_y(\ell)\right]+GD_{xy}\left[v'(\ell)+\theta_x(\ell)\right] +GD_{hx}\left[\theta_z'(\ell)+\psi(\ell)\right]=P_x(\ell) \quad (34)$$

$$GD_{xy}\left[u'(\ell)-\theta_y(\ell)\right]+GD_{yy}\left[v'(\ell)+\theta_x(\ell)\right] +GD_{hy}\left[\theta_z'(\ell)+\psi(\ell)\right]=P_y(\ell) \quad (35)$$

$$EI_{xx}\theta_x'(\ell)=-M_x(\ell) \quad (36)$$

$$EI_{yy}\theta_y'(\ell)=M_y(\ell) \quad (37)$$

$$GJ\theta_z'(\ell)+GD_{hx}\left[u'(\ell)-\theta_y(\ell)\right]+GD_{hy}\left[v'(\ell)+\theta_x(\ell)\right] +GD_{\omega\omega}\left[\theta_z'(\ell)+\psi(\ell)\right]=M_z(\ell) \quad (38)$$

$$EC_w\psi'(\ell)=-M_w(\ell) \quad (39)$$

By substituting into the displacement functions in equation (23), one obtains $\{C_i\}_{12\times1}$ in terms of the end forces

$P_x(\ell)$, $P_y(\ell)$, $M_x(\ell)$, $M_y(\ell)$, $M_z(\ell)$ and $M_w(\ell)$.

Solution for Simply-Supported Composite Beam under End Moments

A simply supported beam with asymmetric section under end moments: bending moments $M_x(z_e)$, $M_y(z_e)$ about X and Y principal axes, and bimoments $M_w(z_e)$ at both ends $(z_e=0,\ell)$ is considered. The end restraints leave the end-sections free to warp and to rotate about X and Y axes. Imposing the following simply supported boundary conditions at member both ends $z=0$ and $z=\ell$:

$$u(0)=0 \quad (40)$$

$$v(0)=0 \quad (41)$$

$$EI_{xx}\theta_x'(0)=M_x(0) \quad (42)$$

$$EI_{yy}\theta_y'(0)=-M_y(0) \quad (43)$$

$$\theta_z(0)=0 \quad (44)$$

$$EC_w\psi'(0)=M_w(0) \quad (45)$$

$$u(\ell)=v(\ell)=0 \quad (46)$$

$$EI_{xx}\theta_x'(\ell)=-M_x(\ell) \quad (47)$$

$$EI_{yy}\theta_y'(\ell)=M_y(\ell) \quad (48)$$

$$\theta_z(\ell)=0 \quad (49)$$

$$EC_w\psi'(\ell)=-M_w(\ell) \quad (50)$$

The closed-form solution for simply supported beam under given end moments is determined by substituting the displacement functions in equation (24) into the above boundary conditions.

NUMERICAL RESULTS AND DISCUSSION

The analytical closed-form solutions developed in the present study are used to investigate the static analysis of thin-walled asymmetric members under general forces. Two examples are conducted for beams with asymmetric channel cross-sections. In both examples, the material is assumed to be steel with $E = 200GPa$ and $G = 77GPa$. Results obtained by the present analytical solutions are compared with (i) Vlasov beam theory which neglects shear deformation and distortional effects, (ii) Abaqus shell S4R element solution with six degrees of freedom per node (i.e., three translations and three rotations) which captures shear deformation and distortional effects.

Example 1: Cantilever under End Transverse Force

A 3.0m cantilever thin-walled beam has an asymmetric J-section is subjected to concentrated transverse force $8.0\,kN$ applied at the corner point A at the cantilever tip (Fig. 2). The centroidal coordinates are; $C_x = 8.205mm$, $C_y = 120.5mm$, coordinates of the shear centre along the principal coordinates are; $x_s = -23.89mm$, $y_s = 42.24mm$, the orientation of principal direction is $\beta = 9.46^o$, and the properties of section are:

$A = 0.78\times10^4 mm^2$, $J = 0.87\times10^6 mm^4$, $I_{xx} = 56.16\times10^6 mm^4$, $I_{yy} = 8.49\times10^6 mm^4$, $C_w = 57.0\times10^9 mm^6$, $D_{xx} = 47.5\times10^4 mm^2$, $D_{yy} = 30.49\times10^4 mm^2$, $D_{xy} = -2.92\times10^4 mm^2$, $D_{hx} = -3.71\times10^4 mm^3$, $D_{hy} = 1.30\times10^4 mm^3$ and $D_{\omega\omega} = 47.14\times10^6 mm^4$.

In the Abaqus shell model, a total of 2,200 S4R shell elements (≈13,940 dof) are used (i.e., 8 elements per upper flange, 4 elements per bottom flange, 10 elements along the web height and 100 elements along the longitudinal axis).

The static analysis results for maximum transverse and lateral displacements u_A , v_A of point A (located at the intersection of the web and upper flange center-lines), bending rotations $\theta_{y\max}$, $\theta_{x\max}$, torsional angle $\theta_{z\max}$ and warping deformation function $\psi_{\max}$ are provided in Table 1. Results based on the present analytical solution are in excellent agreement with results obtained from Vlasov beam solution but slightly differ from those based on Abaqus S4R shell model. This is due to the fact to distortional effects of the cross-section that are captured only in shell element solution, but neither in the present solution nor the Vlasov theory. Due to the non-symmetry of the cross-section, the transverse and lateral responses are fully coupled with twist-warping response, but it is observed that the bending rotation θ_y vanishes in all three solutions.

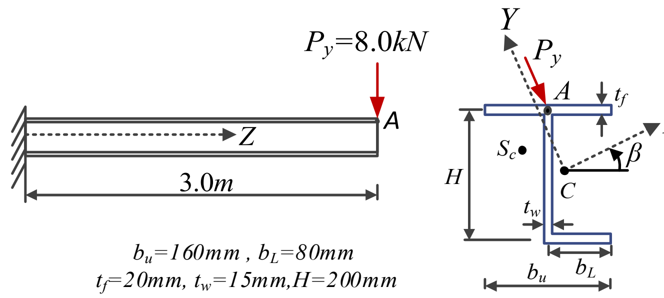
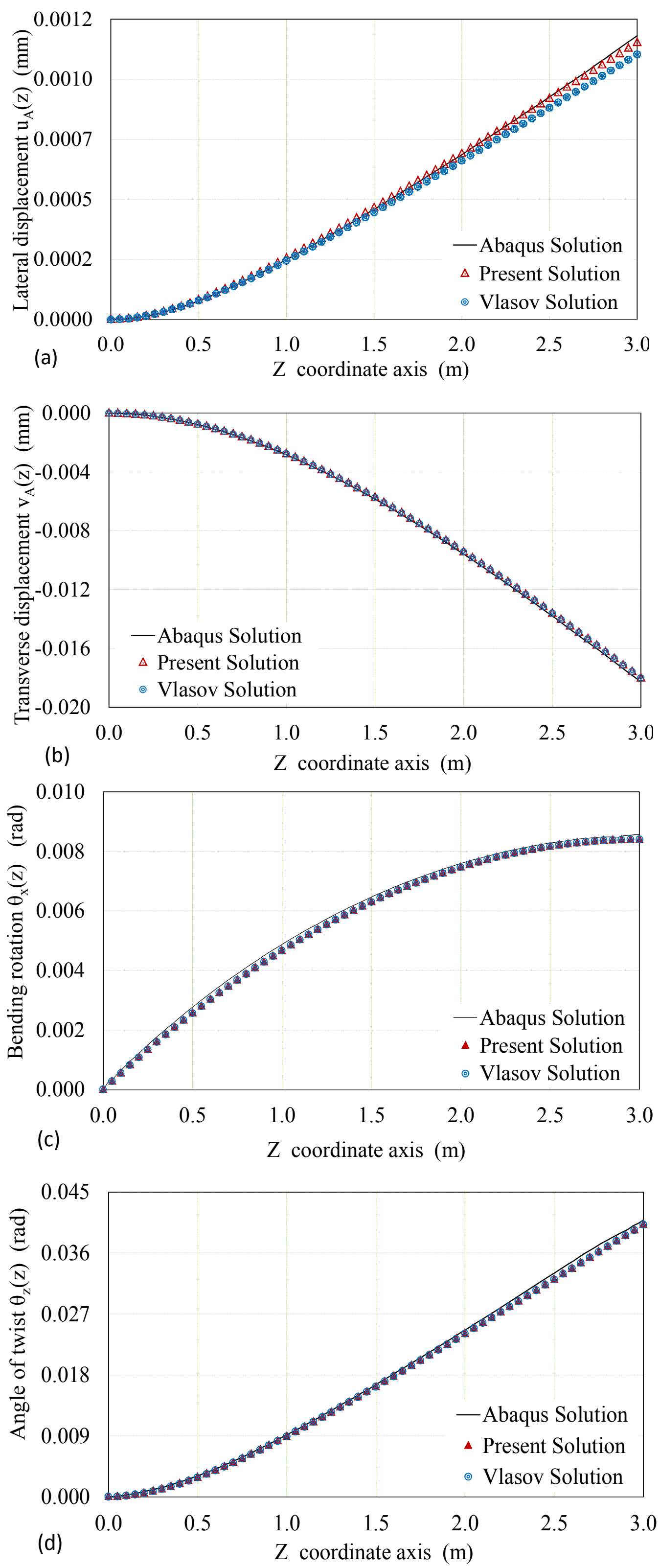


FIGURE 2 A CANTILEVER BEAM UNDER END TRANSVERSE FORCE

TABLE 1: STATIC RESULTS OF COUPLED BENDING-TORSIONAL RESPONSE FOR CANTILEVER J-SECTION

Variable	Abaqus S4R [1]	Present Solution [2]	Vlasov Solution [3]
u_A (mm)	1.133	1.108	1.059
v_A (mm)	-18.30	-18.07	-18.01
θ_x (10^{-3}rad)	8.567	8.408	8.403
θ_z (10^{-3}rad)	40.88	40.25	40.17
ψ (10^{-6} rad/mm)	16.44	16.22	16.16

Figures (3a-f) illustrate the displacements $u(z)$, $v(z)$ and corresponding rotations $\theta_x(z)$, $\theta_y(z)$, twist angle $\theta_z(z)$ and warping deformation $\psi(z)$ along the beam span, respectively. It is observed that the results based on the present formulation coincide with those based on Vlasov beam solution and Abaqus shell model solution except that the lateral displacement slightly deviates from the Abaqus shell solution Again, the differences are attributed to cross-sectional distorsional effects.



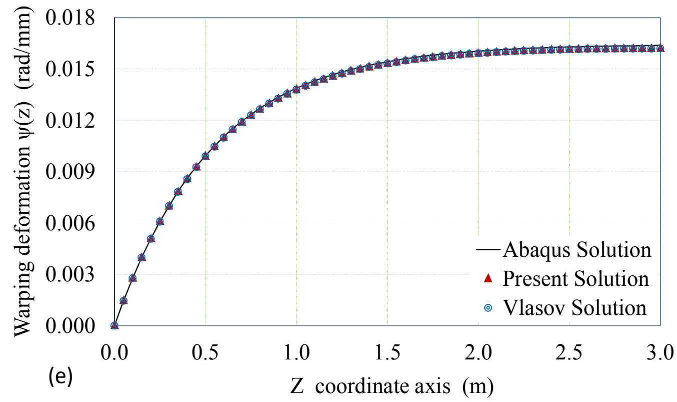


FIGURE 3 STATIC ANALYSIS OF CANTILEVER BEAM UNDER END TRANSVERSE FORCE

Example 2: Cantilever Beam under End Torsion – Shear Deformation Effect

The purpose of this example is to demonstrate the ability of the present solution to capture the shear deformation effects on the coupled transverse-lateral-torsional-warping static response for long span $(\ell=4.0m)$ and short span $(\ell=0.8m)$ cantilevers with asymmetric channel-sections. The long cantilever is subjected to end twisting moment $M_z(\ell)=2.0kNm$ while the short cantilever beam is under the end twisting moment $M_z(\ell)=8.0\,kNm$. The principal coordinates are inclined through an angle $\beta=17.14^o$ (Fig. 4). The coordinates of the centroid in the global coordinate system are $C_x=20mm$ and $C_y=60mm$, while the coordinates of the shear centre S_c along principal axes (X,Y) are; $X_s=-42.83mm$ and $Y_s=-10.29mm$. The properties for the channel-section with respect to the principal coordinate system through the centroid

C are; $A=0.20\times10^4\,mm^2$, $I_{xx}=3.72\times10^6\,mm^4$, $I_{yy}=0.88\times10^6\,mm^4$, $J=0.57\times10^5\,mm^4$, $C_w=0.86\times10^9\,mm^6$, $D_{xx}=11.65\times10^4\,mm^2$, $D_{yy}=8.35\times10^4\,mm^2$, $D_{xy}=1.13\times10^4\,mm^2$, $D_{hx}=-49.15\times10^3\,mm^3$, $D_{hy}=-1.79\times10^2\,mm^3$ and $D_{\omega\omega}=3.22\times10^6\,mm^4$.

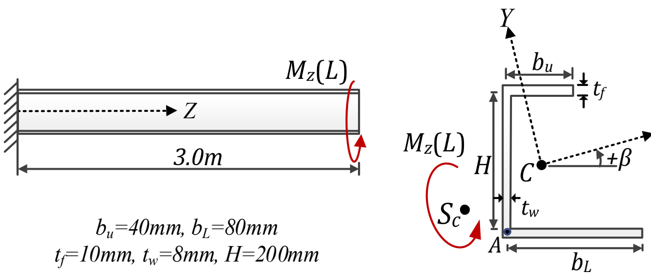
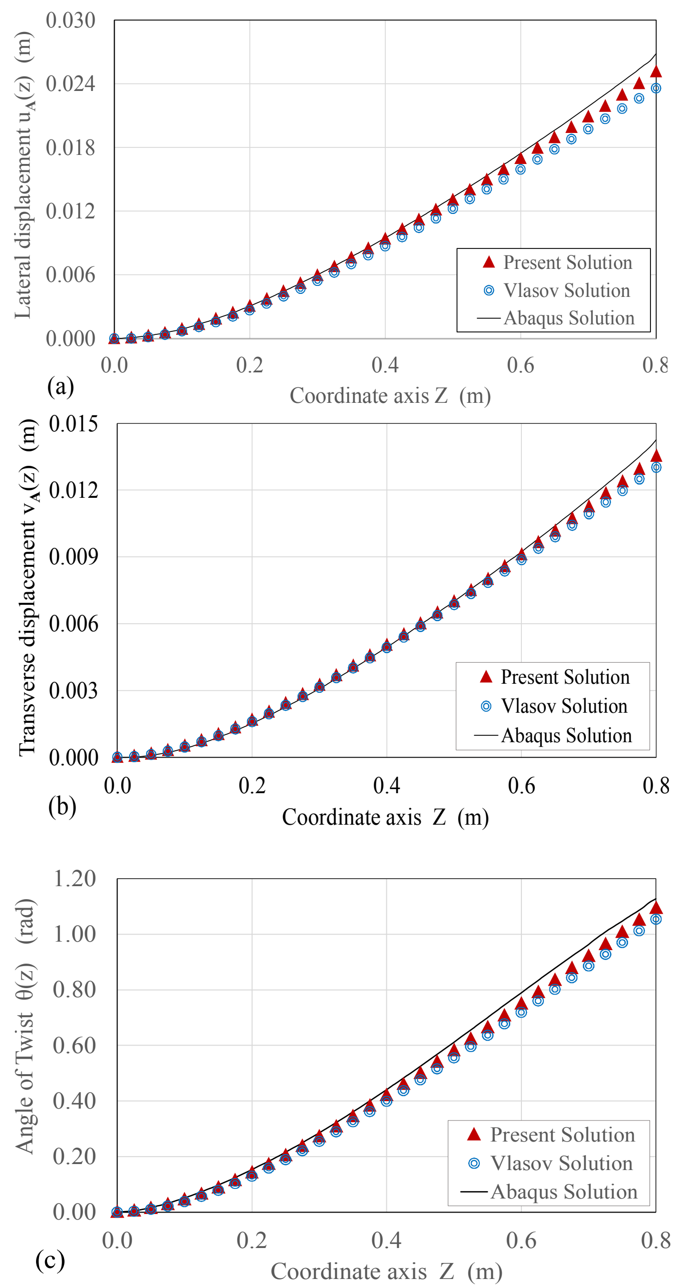


FIGURE 4 CANTILEVER ASYMMETRIC C-SECTION UNDER END TORSION

Three solutions based on; (i) present formulation, (ii) Vlasov beam theory and (iii) Abaqus shell S4R element model for long and short cantilever beams are provided for comparison. In the Abaqus shell solution, the cantilever is subdivided into 50 elements per meter in the longitudinal direction, eight and four elements along the width of top and bottom flanges, and ten elements through the web height. The Abaqus shell models thus consist of 4,400 shell elements for the long span beam and 1,760 shell elements for short span beam.

Table 2 provides the comparisons of maximum lateral displacement $u_{A\max}$, transverse displacement $v_{A\max}$ at point A which located at the intersection of the web and lower flange center-lines (Fig. 4), twist angle $\theta_{z\,\max}$ and warping deformation function $\psi_{\max}$ at the cantilever tip for long and short cantilever beams under end twisting moments. As a general observation, for long cantilever, the static results obtained by the present formulation which captures the shear deformation effects are in excellent agreement with those based on Vlasov beam solution which totally ignores the shear deformation effects and Abaqus shell model which captured the transverse shear deformation and distorsional effects. Under the applied twisting moments, the bending rotation angles $\theta_x(z)$ and $\theta_y(z)$ are observed to vanish in all three solutions, then there is no coupling between bending and twisting deformation (see appendix for more details). To demonstrate the shear deformation effects on lateral-transverse-torsional-warping coupled static deformation, the present study over-predicts the lateral displacement, transverse displacement, twist angle and warping deformation by less than 5.98%, 6.62%, 4.57%, and 5.68% with those based on the Vlasov beam solution. It is noted that shear deformation effects are very significant in short span beams. While due to the inclusion of distorsional effects of the cross-section in Abaqus model, the coupled static results obtained from the present solution are under-predicted 7.17%, 4.41%, 3.11% and 2.93% lower than the corresponding results based on Abaqus shell model. Again, the difference is due to the distortional effects of the cross-section, which are captured only in the Abaqus shell model.

Figures (5a-d) represent static analysis for lateral displacement $u_A(z)$, transverse displacement $v_A(z)$, twist angle $\theta_z(z)$ and warping deformation $\psi(z)$ at the cantilever tip are plotted against the beam axis Z, respectively. It is observed that, the results based on the Vlasov solution in which the shear deformation effects are fully ignored exhibit a significantly less flexible response than those obtained from the present study. Therefore, the present solution static results based on incorporating shear deformations overestimate the coupled lateral-transverse-torsional-warping response when compared with the corresponding results obtained from Vlasov theory.



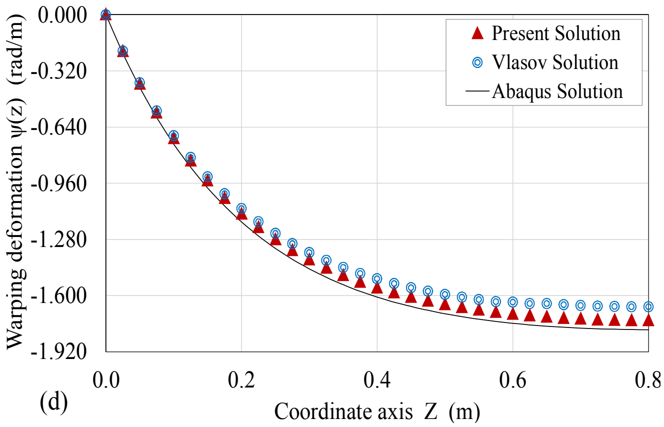


FIGURE 5 STATIC ANALYSIS OF SHORT CANTILEVER ASYMMETRIC C-SECTION UNDER END TWISTING MOMENT

TABLE 2: STATIC ANALYSIS FOR CANTILEVER ASYMMETRIC C-BEAM UNDER END TORSION

Type of beam	Variable	Abaqus S4R	Present Solution	Vlasov Solution
Long beam L=4.0m	u_A (mm)	0.1322	0.1278	0.1273
	v_A (mm)	0.0739	0.0719	0.0715
	θ_z (10^{-3}rad)	1.781	1.738	1.732
	ψ (10^{-6} rad/mm)	-0.4646	-0.4559	-0.4556
Short beam L=0.8m	u_A (mm)	0.0269	0.0251	0.0236
	v_A (mm)	0.0142	0.0136	0.0127
	θ_z (10^{-3}rad)	1.128	1.098	1.044
	ψ (10^{-6} rad/mm)	-1.793	-1.747	-1.643

CONCLUSION

The static equilibrium equations for coupled bending-torsional response and associated boundary conditions for thin-walled beams of asymmetric open cross- sections are derived using the principle of the stationary total potential energy. The present formulation based on a generalized Vlasov-Timoshenko beam theory incorporates the effects of shear deformation due to bending and warping and the bending and twist-warping coupling due to cross section asymmetry. The exact closed-form solutions of coupled equations are obtained for asymmetric beams with cantilever and simply-supported boundary conditions.

The present study effectively captures the coupled bending-torsional static response of thin-walled beams having asymmetric open cross-sections. The numerical results compared with Abaqus shell model and Vlasov beam solutions demonstrate the validity and accuracy of the present formulation. Comparison of the present results with Vlasov beam solution shows the significance of shear deformation effects in short cantilever beams.

NOMENCLATURE

A	Cross-sectional area
b	Length of the flange
C_w	Warping constant
D_{xx}, D_{yy} D_{xy}, D_{hx} $D_{hy}, D_{\omega\omega}$	Section properties
E	Modulus of elasticity
G	Shear modulus
$h(s)$	Normal distance between the shear centre and the tangent to mid-surface
H	Height of beam cross-section from the flanges mid-surfaces
I_{xx}, I_{yy}	Moment of inertias of the cross-section about the principal X and Y axes
J	Torsional constant
ℓ	Length of the member
$M_j(z)$	Concentrated moment about j-th direction (for $j = x, y, z$)
$M_w(z)$	Concentrated bimoment
$m_j(z)$	Distributed moments about j-th direction (for $j = x, y, z$)
$m_w(z)$	Distributed bimoment
n, s, z	Local curvilinear coordinate system
P_z	Concentrated force along longitudinal axis
$q_j(z)$	Distributed forces along x, y, z directions (for $j = x, y, z$)
S_c	Shear centre of the cross-section
t_1, t_2	Time intervals
T^*	Kinetic energy
$\bar{u}, \bar{v}$	Displacements of the shear centre along the principal X, Y axes
U^*	Internal strain energy
$\bar{V}_j(z)$	Shear forces along x, y axes (for $j = x, y$)
$\bar{w}$	Average longitudinal displacement along the Z axis
W^*	Work done by applied forces
X, Y, Z	Principal coordinate system

$x(s), y(s)$	Coordinates of arbitrary point on mid-surface of the section along X and Y axes
ρ	Density of the material
r_o	Polar radius of gyration
$\theta_x, \theta_y, \theta_z$	Rotations angles around the X, Y, Z axes, respectively
$\hat{\alpha}(s)$	Angle between the tangent to the cross-section and the principal X axis
$\bar{\psi}$	Warping deformation function
Ω	Exciting frequency
$\omega(s)$	Warping function of the cross-section

REFERENCES

[1] N. Bercin and M. Tanaka, Finite element modeling of the coupled bending and torsional free vibration of uniform beams with an arbitrary cross-section, Applied Mathematical Modelling, 21(6), (1997), 339-344.

[2] M. Y. Kim, N. Kim and H. T. Yun, Exact dynamic and static stiffness matrices of nonsymmetric thin-walled beam-columns, Computers and Structures, 81(14), (2003), 1425-1448.

[3] N. Kim and M. Y. Kim, Exact Dynamic/Static Stiffness Matrices of Non-symmetric Thin-Walled Beams considering coupled shear deformation effects, Thin-walled Structures, 43 (5), (2005), 701-734.

[4] J. Li, W. Li and H. Hua, Coupled bending and torsional vibration of non-symmetrical axially loaded thin-walled Bernoulli-Euler beams, Mechanics Research Communications, 31(6), (2004), 697-711.

[5] A. Prokic, On fivefold coupled vibrations of Timoshenko thin-walled beams, Engineering Structures, 28(1), (2006), 54-62.

[6] T. P. Vo and J. Lee, On six-fold coupled buckling of thin-walled composite beams, Composite Structures, 90(3), (2009), 295-303.

[7] S. N. Jung and J. Y. Lee, Closed Form Analysis of Thin Walled Composite I-Beams considering non-classical Effects, Composite Structures, 60, (2003), 9-17, 2003.

[8] N. Kim, B. J. Lee, and M. Kim, Exact element static stiffness matrices of shear deformable thin-walled beam-columns, Thin-walled Structures, 42(9), (2004), 1232-1256.

[9] D. Ambrosini, On free vibration of non-symmetrical thin-walled beams, Thin- Walled Structures, 47(6-7), (2009), 629-636.

[10] F. de Bordon and D. Ambrosini, On free vibration analysis of thin-walled beams axially loaded, Thin-Walled Structures, 48(12), (2010), 915-920.

[11] D. Ambrosini, Experimental validation of free vibrations from non-symmetrical thin walled beams, Engineering Structures, 32(5), (2010) 1324-32

[12] V. Vlasov, *Thin-walled elastic beams*, Jerusalem, Israel Prog. for Scientific Translation, 1961.

Appendix: Proof That Bending Rotations Vanish for a Cantilever with No External Forces

It is required to formulate the expression for the rotation angles $\theta_x(z)$ and $\theta_y(z)$ for a cantilever beam with no externally distributed lateral force $q_x(z)$ and bending moment $m_y(z)$. By setting the right hand side of the static equilibrium equations related to lateral response (Eqs. 10 and 13) equal to zero, one obtains:

$$G\left[D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)+D_{hx}\left(\theta_z'+\psi\right)\right]'=0 \quad \text{(A1)}$$

$$\begin{aligned}-EI_{yy}\theta_y'' - G\Big[D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)&\\ +D_{hx}\left(\theta_z'+\psi\right)\Big]&=0\end{aligned} \quad \text{(A2)}$$

and the related boundary conditions are:

$$\begin{aligned}\Big[G\left\{D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)+D_{hx}\left(\theta_z'+\psi\right)\right\}&\\ -V_x(z)\Big]\delta u(z)\Big|_0^{\ell}&=0\end{aligned} \quad \text{(A3)}$$

$$\left[EI_{yy}\theta_y' - M_y(z)\right]\delta\theta_y(z)\Big|_0^{\ell}=0 \quad \text{(A4)}$$

For simplicity, equation (A1) can be re-write as:

$$\begin{aligned}\chi'(z)=-G\Big[D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)&\\ +D_{hx}\left(\theta_z'+\psi\right)\Big]'&=0\end{aligned} \quad \text{(A5)}$$

Integrating equation (A5) with respect to z, leads to:

$$\begin{aligned}\chi(z)=-G\Big[D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)&\\ +D_{hx}\left(\theta_z'+\psi\right)\Big]&=C_1\end{aligned} \quad \text{(A6)}$$

From equation (A6), by substituting into (A2), one obtains:

$$-EI_{yy}\theta_y''(z)-\chi(z)=0 \quad \text{(A7)}$$

For the cantilever beam, the boundary conditions at the fixed and free end $z=0$:

$$u(0) = 0 \quad \text{and} \quad \theta_y(0) = 0 \tag{A8}$$

and at the cantilever free end $z = \ell$, noting that no external bending moment $M_y(\ell)$ nor shear force $V_x(\ell)$ are applied, Equations (A3) and (A4) are:

$$EI_{yy}\theta_y'(\ell) = 0 \tag{A9}$$

$$G\left[D_{xx}\left(u' - \theta_y\right) + D_{xy}\left(v' + \theta_x\right) + D_{hx}\left(\theta_z' + \psi\right)\right]_{z=\ell} = 0 \tag{A10}$$

From equation (A6), by substituting into (A10), one obtains:

$$\chi(\ell) = 0 \tag{A11}$$

From equation (A6), one obtains:

$$\chi(z) = \chi(\ell) = 0 \tag{A12}$$

or $C_1 = 0$. From (A12) by substituting into equation (A7), and by integrating the resulting equation, one obtains:

$$\theta_y(z) = C_2 z + C_3 \tag{A13}$$

Imposing the boundary conditions in equations (A8) and (A9) into (A13), yields:

$$C_2 = C_3 = 0 \tag{A14}$$

From equation (A14), by substituting into equation (A13), leads to conclude that the bending rotation about Y axis is vanished, i.e.,

$$\theta_y(z) = 0 \tag{A15}$$

In a similar manner, the other bending rotation angle $\theta_x(z)$ is also found to vanish, i.e.,

$$\theta_x(z) = 0 \tag{A16}$$

In summary, for the special case of static response of a cantilever beam with asymmetric section, the bending rotations $\theta_x(z)$ and $\theta(z)$ for the member are shown to vanish.