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STABILITY ANALYSIS OF PLANE COUETTE-POISEUILLE FLOW WITH POROUS WALLS UNDER TRANSVERSE MAGNETIC FIELD

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ABSTRACT

A linear stability analysis of a plane Couette-Poiseuille flow of an electrically conducting fluid with uniform cross-flow is investigated in the presence of a transverse magnetic field. The Chebyshev spectral collocation method is utilized to obtain eigenvalues of the modified classical Orr-Sommerfeld equation. The effect of cross-flow with its sense and transverse magnetic field on the stability are examined. The results show that cross-flow acts to stabilize or destabilize the flow. The cross-flow's sense produces a significant influence on the stability. This effect becomes more important in the presence of a magnetic field. The dependence of the magnetic field's effect on the cross-flow's sense is also presented.

INTRODUCTION

Hydrodynamic stability of flow between two porous parallel plates has long been important in most industries, especially in the biomedical industry, filtration systems and environmental engineering [1–7]. In this way, serval works and investigations have been devoted to control the flow's instability. For instance, Hains [8] studied the effect of cross-flow (blowing/suction) on the stability of a plane Poiseuille and Couette-Poiseuille flows. He found that, a modest amount of uniform injection/suction of the same fluid produced a significant increase in the critical Reynolds number, and in the plane Poiseuille flow case, it destroyed the velocity profile's symmetry. In addition, he examined the influence of cross-flow's sense and showed that the flow is more stable when the fluid is injected at the stationary wall. A similar study has been carried out for the flow between parallel porous stationary walls by Sheppard [9]. In this investigation [9], the author has defined two Reynolds numbers related respectively to, the maximum symmetric plane Poiseuille velocity (without cross-flow) and the cross-flow velocity. Hereafter, he has confirmed numerically, using the Galerkin method, the results given by Hains [8], also he has compared the results to the sufficient condition for stability established by [10]. Recently, Fransson and Alfredsson [7] have studied, in details the hydrodynamic stability of plane Poiseuille flow subject to a uniform cross-flow in which they have made corrections to previous works [8, 9]. In particular, they separated the effects of the velocity distribution from those of the magnitude of the velocity in the basic state, by using the maximal channel velocity of plane Poiseuille flow with the presence of a cross-flow as their characteristic velocity. In addition, they have proved that this problem depends on the choice of the characteristic velocity. Furthermore, they found the stabilizing and the destabilizing effect of the cross-flow. An extension of this work [7] has been given by Guha and Frigaard [11], who studied the stability of plane Couette- Poiseuille flow with uniform cross-flow (the injection in the stationary wall and the suction to the moving wall). The authors have obtained an exact basic solution defined by different expressions for each range of CR_C (C, is the velocity ratio of the wall and classical Poiseuille velocities, R_C represents the cross-flow Reynolds number). Their results are in good agreement with those of Fransson and Alfredsson [7] in the case of plane Poiseuille flow. They have also shown that the wall's velocity has a stabilizing effect on the flow that became unconditionally stable at (Re, α) = (6000, 1) for C = 1.

In the other hand, numerous studies have been devoted to illustrate the effect of an external uniform transverse magnetic field on the channel flow's stability when the considered fluid is assumed to be electrically conducting [12-20]. For highly electrically conducting fluids, i.e., very small magnetic Reynolds number, R_m, compared to unity, the governing equations of the plan Poiseuille flow's stability were simplified by Lock [12]. He found, by using an asymptotic method, that the magnetic field has a stabilizing effect. Since then, this investigation has attracted much attention and has been extended and developed in many ways. As example, we cite the Takashima's work [13], in which he has reexamined the Lock's problem [12]. He focused his study to analyze the effect of a transverse magnetic field on the stability of the modified plane Poiseuille flow without Lock's simplification [12]. In this study, author shown that except for the case where the magnetic Prandtl number, P_m (= R_m/Re), is sufficiently small, the magnetic field has a stabilizing effect. In other papers, the hydromagnetic stability of the plane Couette flow was treated by Kakutani [14]. He neglected the interaction of the magnetic field and the velocity perturbation and it was found that, in outside the range [0; 3.91[the flow is linearly stable in the Couette flow case. This unstable range has been more accurately reduced ([0; 3.88[) by Takashima [16], using a Chebyshev collocation method.

In the present paper, we extend the previous works studied in [8, 11] to include the influence of a uniform transverse external magnetic field on the hydrodynamic stability of a plane Couette-Poiseuille for an electrically conducting viscous incompressible fluid. The physical motivation of this work is to assess how the cross-flow (with its sense) and the magnetic field interact to affect the hydrodynamic stability of a flow. Our used approach in this investigation will be based on the analysis the least stable eigenmode of the most dangerous mode (fundamental mode, α=1), this can allow us to sweep a very large range of the physical parameter's values of the stability problem. This paper is organized as follows. The model physique is described and the basic flow solutions are given in the second section. The perturbed problem is reduced to a modified classical Orr-Sommerfeld equation. Then the numerical resolution method of this equation and the results that validate our numerical code are presented in the third section. The pertinent results are discussed qualitatively as function to various physical parameters of the stability problem in the fourth section. Finally, the sixth section concludes our paper.

PHYSICAL MODEL AND LAMINAR SOLUTIONS

Consider a plane Couette-Poiseuille flow of an electrically conducting viscous incompressible fluid between two porous parallel plates separated by a fixed distance 2d. The cross-flow of constant velocity, nv_o, is applied to the fluid in the transverse direction, $\mathbf{y}^*$, with the presence of a uniform external magnetic field, H_O, parallel to the $\mathbf{y}^*$-axis. The upper plate at $y^* = +d$ is moving in the $\mathbf{x}^*$ direction with a uniform velocity, U_w, and the other one, at $y^* = -d$, is stationary. We assume that the external electric field is zero, the induced magnetic field is negligible, the magnetic Reynolds number, R_m, is very small [17] and the plates are electrically non-conducting. Two configurations are envisaged, the first one concerns an injection at lower plate and suction at upper plate (first case, n = 1). The second configuration consists in an injection at upper plate and suction at lower plate (second case, n = −1). The mathematical equations modelling the flow in their dimensional forms (*) are, respectively, the continuity equations, the modified Navier-Stokes equation and the induction equation:

$$\nabla . V^* = \nabla . H^* = 0 \qquad (1)$$

$$\rho\left[\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^* . \nabla \mathbf{V}^*\right] = -\nabla P^* + \rho \upsilon \Delta \mathbf{V}^* + \frac{\mu}{4\pi}\left[rot\mathbf{H}^* \wedge \mathbf{H}^*\right] \qquad (2)$$

$$\frac{\partial \mathbf{H}^*}{\partial t^*} - rot\mathbf{V}^* \wedge \mathbf{H}^* = \lambda \Delta \mathbf{H}^* \qquad (3)$$

where ρ, ν, μ, λ, P^* and $\mathbf{V}^*$ are, respectively, the fluid density, the kinematic viscosity, the magnetic permeability, the magnetic diffusivity, the pressure and the velocity. Further, H^* is the magnetic field strength.

The boundary conditions at the walls ($y^* = \pm d$) are:

$$V^*_{x^*} = 0 \;\; at \;\; y^* = -d \; ; V^*_{x^*} = U_w \;\; at \;\; y^* = d$$

$$H^*_{x^*} = 0 \;\; at \;\; y^* = \pm d \qquad (4)$$

Using reference variables d, $V^*_{\max}$, d $(V^*_{\max})^{-1}$, ρ $(V^*_{\max})^2$ and H_O for, respectively, length, velocity, time, pressure and magnetic field, the basic velocity profile in non-dimensional form can be written as:

$$V_x(y) = \begin{cases} \dfrac{(4\sinh(b) - abCe^{-b})(e^{ay} - \cosh(a)) - (4\sinh(a) - abCe^{-a})(e^{by} - \cosh(b)) + 4abC\sinh(b-a)}{abC(e^{-a+bY} - e^{-b+aY}) + 4\sinh(b)(e^{aY} - \cosh(a)) - 4\sinh(a)(e^{bY} - \cosh(b))} & C > C_o \\ \\ \dfrac{abC(e^{-a+by} - e^{-b+ay}) + 4\sinh(b)(e^{ay} - \cosh(a) - 4\sinh(a)(e^{by} - \cosh(b))}{2abC\sinh(b-a)} & C \le C_o \end{cases} \qquad (5)$$

with

$$Y=\frac{b}{(a-b)a}\log\left(\frac{(4+abC)\sinh(a)-abC\cosh(a)}{(4+abC)\sinh(b)-abC\cosh(b)}\right) \quad (6)$$

$$C_O=2\frac{(be^{a}-ae^{b})(be^{-a}-ae^{-b})^{-1}-1}{ab} \quad (7)$$

$$a,b=\frac{1}{2}\left[R_C^2\pm\sqrt{R_C^2+4M^2}\right] \quad (8)$$

This basic flow depend on the cross-flow Reynolds number, R_C, the Hartman number, M, and the velocity ratio, C, defined as:

$$R_C=\frac{v_o d}{\upsilon} \qquad M=\mu d H_o\sqrt{\frac{\sigma}{\rho\upsilon}} \qquad C=\frac{U_W}{U_P} \quad (9)$$

where σ is the electrical conductivity and U_P is the maximum velocity of the classical plane Poiseuille flow.

LINEAR STABILITY ANALYSIS AND NUMERICAL METHOD

In order to study the linear stability of this problem, we use the Squire theorem [25]. The infinitesimal perturbations (**v**, **h** and p) are superimposed to the basic flow variables (**V**, **H** and P). Then the solutions can be sought into Fourier's modes as follows:

$$(v,h,p)=\left[\psi(y),\,\phi(y),p_1(y)\right]e^{i(\alpha x-\alpha ct)} \quad (10)$$

where **ψ**(y), **φ**(y) and p_1(y) are, respectively, the complex amplitudes of the perturbations (**v**, **h** and p), α is the wave number, c is the complex wave speed and $i^2 = -1$. Applying the Lock's assumption [12], the differential equation determining the stability is expressed by:

$$(V_x-c)(\psi,_{yy}-\alpha^2\psi)-V_x,_{yy}\psi+\frac{i}{\alpha\,\mathrm{Re}}(\psi,_{yyyy}-2\alpha^2\psi,_{yy}+\alpha^4\psi)=\frac{iM^2}{\alpha\,\mathrm{Re}}\psi,_{yy}+\frac{inR_C}{\alpha\,\mathrm{Re}}(\psi,_{yyy}-\alpha^2\psi,_y) \quad (11)$$

Equation (11) represents the modified Orr-Sommerfeld equation in which added two additional terms. The first term is due to the cross-flow (R_C), in the case corresponding to Rc= 0, equation (9) reduces to the stability equation of Takashima [13]. The second one reflects the effect of an external magnetic field (M), when this term is zero, Eq. (11) is similar to that given by Guha and Frigaard [11]. The correspond boundary conditions are

$$\psi(y=\pm1)=\psi,_y(y=\pm1)=0 \quad (12)$$

Re is the Reynolds number defined as:

$$\mathrm{Re}=\frac{V^*_{\max}d}{\upsilon} \quad (13)$$

The governing stability equations, (11) and (12) are solved numerically using the Chebyshev spectral collocation method based on N collocation points of Gauss- Labatto [26,27]. Under these conditions, our stability problem is reduced to an algebraic system with c (= $c_r + ic_i$) eigenvalues:

$$E\psi=cF\psi \quad (14)$$

where **E** and **F** are two matrices that depend on *M, Re, R_C , C, α* and *N*. Note that, the flow is linearly unstable if c_i >0.

To test the calculation code, we will check the results obtained by Takashima [13], when *M* is no-zero and no-cross-flow with *C* = 0, also with Fransson and Alfredsson [7] when no magnetic field is applied in the presence of a cross-flow with *C* = 0. These results are represented in tables (1) and (2), and they are in excellent agreements with those obtained by these authors [7, 13]. Also, we note that our results are in an excellent agreement with those of Guha and Frigaard [11, table 2] in the case of *$R_C \neq 0$, $C \neq 0$* and *M = 0.*

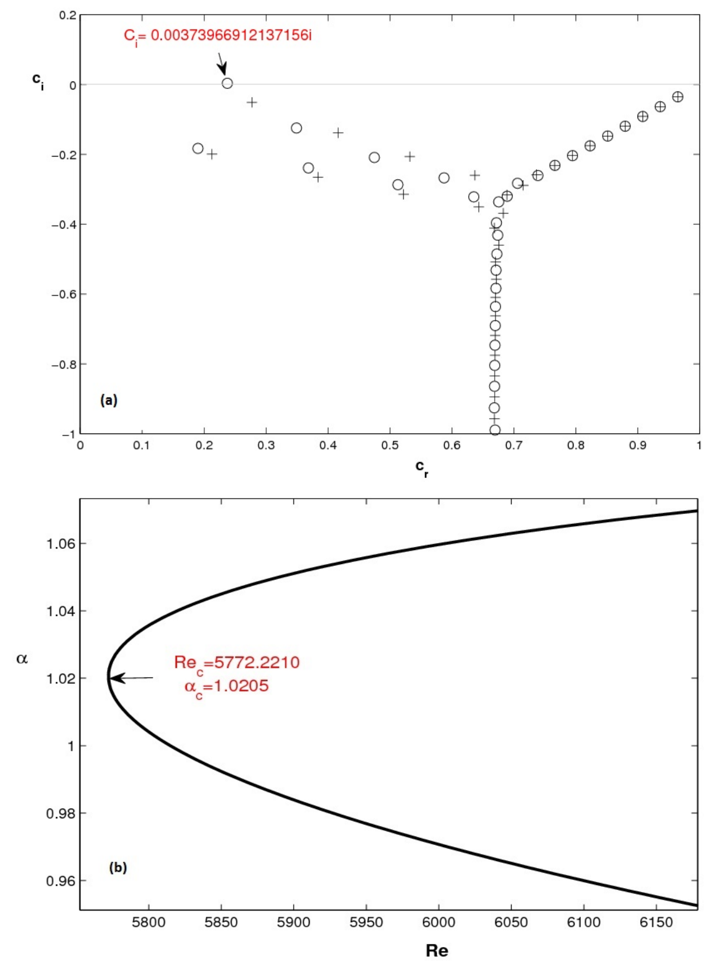

Figure 1 (a) The imaginary part c_i as function c_r at (*Re, α, R_C, C, M*) = (10000, 1, 0, 0, 0) for *n* = ±1, odd mode (+) and even mode (o). (b) Neutral stability curve at (*R_C, C, M*) = (0, 0, 0) for *n* = ±1

Table 1 Critical Reynolds number Re_c and wave number α_c versus cross-flow Reynolds number R_C for M = 0, C = 0 and n = ±1

	Our results		Fransson and Alfredsson [7]	
R_C	Re_c	α_c	Re_c	α_c
0	5772.2210	1.0205	5772.22	1.02039
0.2	5966.798	1.0132	5967.01	1.01189
0.4	6607.2664	0.9913	6607.4	0.99025
0.6	7902.2166	0.9551	7902.5	0.95361

Table 2 Critical Reynolds number Re_c and wave number α_c versus the Hartmann number M for R_C = 0 and C = 0.

	Our results		Takashima [13]	
M	Re_c	α_c	Re_c	α_c
0	5772.2210	1.0205	5772.2218	1 .020547
0.5	6706.0912	1.0057	6706.0911	1.005734
1	10016.2622	0.9718	10016.2621	0.971828
3	65155.2127	0.9582	65155.210	0.958249
5	164089.997	1.1342	164089.994	1.134248
10	439818.147	1.739	439818.16	1.73915
15	708952.511	2.457	708962.18	2.45660
20	961806.807	3.2290	961767.17	3.23764

For the no-cross flow without magnetic field and for $C = 0$, we have computed the eigenvalues for $Re = 10000$, $\alpha = 1$ and $N = 60$. The obtained results are complete agreement with the list of Orszag [22, table 5]. The spectrum is plotted in figure 1 (a). It is worth noting, from the results in this figure, that the curves converge in excellent agreement toward that obtained by Dongorra et *al*. [23] when they used the Chebyshev tau-QZ algorithm method and Hifdi et *al*. [24] and Rafiki et *al*.[19] by the Chebyshev spectral collocation method. Moreover, the critical Reynolds and wave numbers are converging, respectively, to Rec = 5772.2210, α_c = 1.0205 [22] for 60 collocation points [see figure 1 (b)].

RESULTS AND DISCUSSION

The cross-flow's sense effect on the least stable eigenmode of the most dangerous mode, α = 1, is shown in Fig. 2, for Re = 6000 at M = 0. The largest imaginary part, c_{imax}, is a function of cross-flow Reynolds number, R_C: as C equal zero (— •), corresponds to the plane Poiseuille flow, ci decreases from c_{imax}=0.0003231 for R_C = 0 to c_{imax}=-0.0477 for R_C = 3.48. Then, increases until c_{imax}= 0.03844 at Rc = 276.6. From this value of R_C, c_{imax} starts to decrease monotonically as a function of R_C and the flow becomes again stable from $R_C \geq 636.20$. It is also important to note that the evolution of c_{imax} remains the same in both studied cases, n=±1. This remark indicated that the cross-flow's sense has no effect on the stability of plane Poiseuille flow.

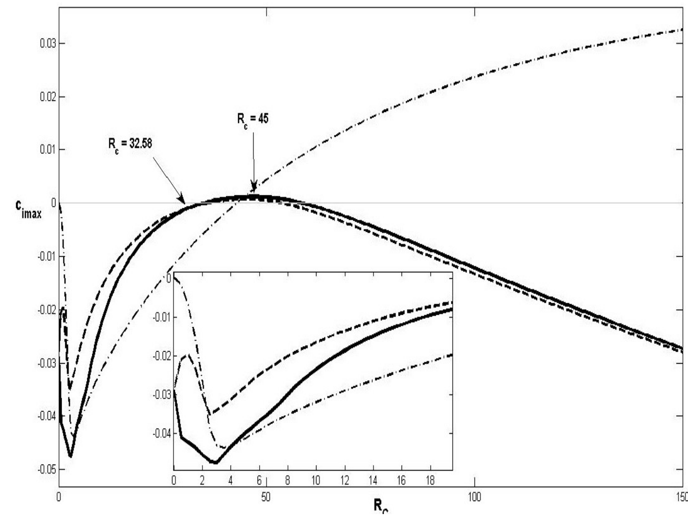

Figure 2 The effect of the sense of cross-flow (R_C) on the least stable imaginary part c_{imax} of the most dangerous mode at (Re , α , C , M)= (6000 , 1 , 0.5 ,0) for the both case n = 1 (—)[11]and n = −1 (— —). With C = 0 for n=±1 (— •) [7]

For the non-zero C corresponds to a plane Couette-Poiseuille flow. The curves (—) and (— —) represent, respectively, the first and the second case, n = ±1, which are obtained for C = 0.5. It is evident from these curves that the wall's movement tends generally to stabilize the flow and reduce the unstable range of cross-flow Reynolds number [11]. As the sense of cross-flow is changed, we observe that a significant effect on the evolution of the largest imaginary part: c_{imax} can be weakly increased or decreased as Rc increases. Indeed, the flow is more stable in the case which corresponds to when the fluid is injected in the stationary wall, n = 1, than wherein the fluid's injection takes place in the movable wall, n = -1. This effect becomes reversed from $R_C > 32.58$. In this situation, the position of maximum streamwise velocity is displaced toward the fixed that leads to detract the moving wall effect when R_C is very large.

In Figs. 3 and 4 the effect of magnetic field on the flow's stability in the both cases, n=±1, by varying the Hartman number, keeping the wall velocity constant is analyzed. The results are obtained for C=0.5, α = 1 and Re = 6000. By inspecting Fig.3, we can see how the magnetic field and cross flow interact (when they have the same sense, n=1) to influence the evolution of the least stable imaginary part c_{imax}. Comparing our results with those of Guha and Frigaard [11] [curve (—)], the largest value of c_{imax}, corresponding to the maximum of destabilization, decreases with an increase of Hartmann number that leads to stabilize the flow. The cross-flow Reynolds number, for which the largest imaginary part c_{imax} is maximized, decreases with the Hartmann number. The range of R_C corresponding to a destabilizing effect narrows as M increases. These results suggest that the instability caused by cross-flow is well controlled by including an external magnetic field.

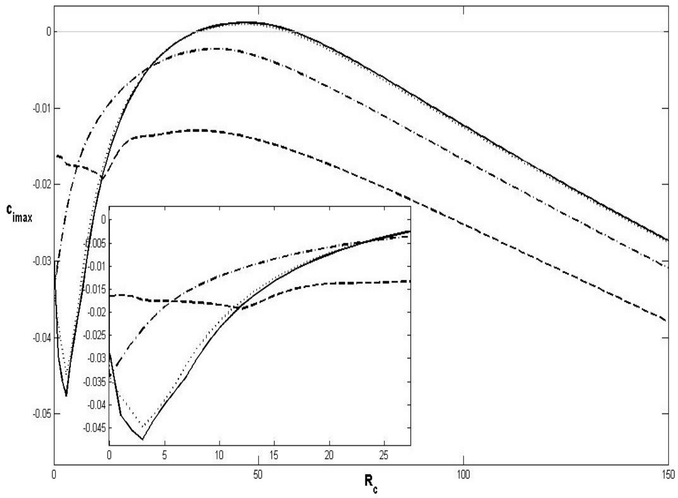

Figure 3 The effect of R_C on the least stable imaginary part c_{imax} of the most dangerous eigenmode at *(Re, α, C)* = (6000, 1, 0.5) and different values of Hartmann number, M≈0 (—), M = 1 (••), M = 4 (— •) and M = 7 (— —) for n = 1

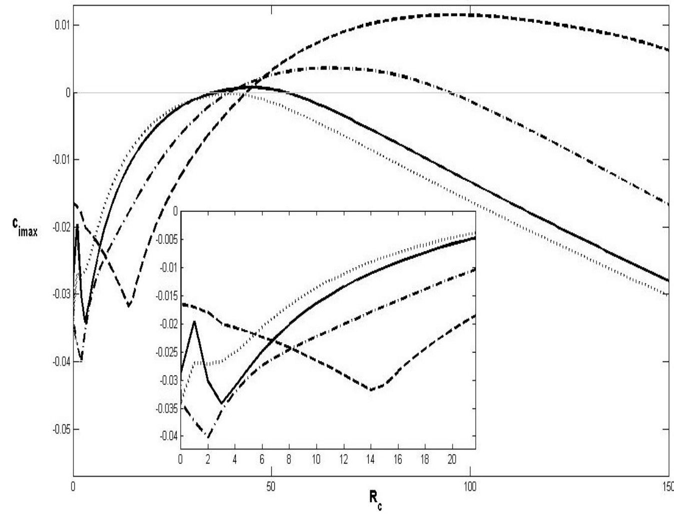

Figure 4 The effect of R_C on the least stable imaginary part of the most dangerous eigenmode at (Re, α, C) = (6000, 1, 0.5) and different values of Hartmann number, M=0 (—), M = 1 (••), M = 4 (— •) and M = 7 (——) for n = −1.

Fig. 4, exhibits the evolution of c_{imax} as a function of R_C when the cross-flow and the magnetic field have the opposite sense, n=-1. In contrast to the curve corresponding to M=1, in which the stabilizing effect is always present, the curves corresponding to M=4 and 7 give rise to either a stabilizing or destabilizing effect depending to the value of R_C. The cross-flow Reynolds number corresponding to the largest value of c_{imax} decreases from R_C=90 for M=7 to R_C=40 for M=1 and increases to the value R_C = 45 for M=0. The largest value of c_{imax} in which the destabilization is maximized, initially decreases and then increases with increasing M. Furthermore, a certain value of Hartman number can exist, for that the flow is linearly stable, i.e., c_{imax}<0. This result can be further confirmed in Fig.5. In addition, the magnetic field tends to expand the unstable range of cross-flow Reynolds number by varying the Hartmann number. Here, it turns out that the magnetic field has both stabilizing and destabilizing effect. This effect is apparently caused by the interference between the waves occurred by cross-flow, the wave generated by the moving wall and produced by magnetic waves.

Fig. 5 depicts the least stable imaginary part, c_{imax}, versus the Hartmann number for the both cases, n=±1. We fix C=0.5, α = 1, Re = 6000 and the cross-flow Reynolds number at R_C=45, which corresponds to the most unstable value of c_{imax}. It can be seen from Fig. 5 that, for n=1, c_{imax} decreases dramatically as the Hartmann number increases from M=0 to M=20, and for M>2.42, the flow is unconditionally stable (c_{imax}< 0). In the second case, (n= -1), c_{imax} initially passes from 0.0007363 to -0.01276 when M increases from 0 to 2 and then increases with increasing Hartmann number. The flow turn into an unstable situation at M=2.96. Form M>4.703, the value of c_{imax} starts to decrease again and re-stabilizes at M= 7.583. Thereafter, the curve representing the c_{imax}-evolution starts asymptotically to converge towards that of the first case, n=1, from M=13.3. In addition, the results in Fig. 5 indicate, by comparing the evolution of c_{imax} in the both cases, that the flow's stability is very sensitive to the cross-flow's sense when the magnetic field is present, tends to stabilize in the first case, n=1, and also destabilize in the second case, n=-1. We notice that when Hartmann number is sufficiently large (M>18.2, approximatively), the cross-flow's sense has no significant effect on the variation of c_{imax}. Indeed, the onset suppression of instability caused by the cross-flow's sense is apparently due to the generated wave by the magnetic field which is dominant than that produced by cross-flow.

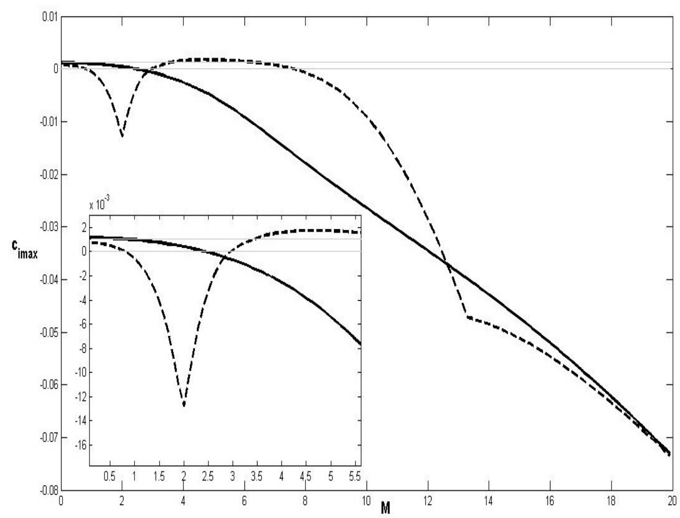

Figure 5 The effect of *M* on the imaginary part c_i of the most dangerous eigenmode at *(Re, α, C, R_C)* = (6000, 1, 0.5, 45) for the both configurations n = 1 (—) and n = −1 (— •)

CONCLUSION

We have studied qualitatively the linear stability analysis of a plane Couette-Poiseuille flow, which is maintained under the combination of a uniform cross-flow and an external uniform magnetic field. The linear problem has been reduced to a modified classical Orr-Sommerfeld equation. We have focused our analysis on the effect of the cross-flow Reynolds number, R_C, the cross-flow's sense, n= ±1, and that of the Hartmann number, M on the behavior of the least stable eigenmode of the fundamental mode (α=1). It was shown that an increase in R_C leads to stabilize or destabilize the flow in the both cases (n= ±1). In contrast to the plane Poiseuille flow where the sense of cross-flow has no effect on the flow stability, the Couette-Poiseuille flow is sensitive to this change of the sense. A critical value of cross-flow Reynolds number may exist in which the flow becomes least stable in the first case, n= 1, than the second one, n= -1. This effect can be suppressed in the presence of an external magnetic field when Hartmann number is sufficiently large. In addition, the magnetic field has a stabilizing effect in the first case, n= 1, and gives rise to either a stabilizing or destabilizing effect in the second one.

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